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History of Mathematics: Picking up Triangles

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Euclid's *Elements* [of Geometry] is one of the most influential books ever written. It was first compiled in about 300 BCE, and it reigned supreme in the classroom into living memory. When I first studied geometry, I learned it from a text that in very large measure followed Euclid. Beyond this impressive record of endurance, it very much defined the mathematical agenda. It began with **Definitions** of the objects to be discussed (points, lines, angles, triangles, etc.), went on to **Common Notions** or **Axioms,** and to **Postulates**. The common notions were assumptions, but general ones, not specifically geometric in nature. For example: Things which are equal to the same thing are equal to each other.

The postulates were more specifically geometric in nature. There were five in all. The first three refer to the possibility of "construction":

1. It is possible to draw a straight line connecting any point to any other point.

2. It is possible to extend a finite straight line continuously in a straight line beyond its original end-points.

3. It is possible to draw a circle with any given centre and of any [given] radius.

The fourth postulate (of which more later) states that *all right angles are equal*. The fifth is the notorious Parallel Postulate. This has generated an enormous amount of literature, and I wrote a column on it in *Function* in October 1999.

Following the definitions, the common notions and the postulates come the **Propositions**, nowadays called either Theorems or Constructions. The theorems, in our modern nomenclature, are statements that can be proved from the definitions, common notions, postulates and those theorems preceding them in the overall schema. The constructions show how (with ruler and compasses) we may make the diagrams that illustrate the propositions. (But constructions also require proof: we need to be assured that they work!) The entire enterprise is seen as typical of mathematics as a whole in that an elaborate and impressive array of knowledge is erected on a very small base.

The propositions are numbered, and the one I want to begin with is that referred to as Proposition I.4, that is to say Proposition 4 of the first book (of the 13 that make up the entire work). Propositions I.1, I.2 and I.3 are all what in modern terms we would call constructions. I.2, for example, gives a recipe by means of which, using ruler and compasses, a straight line AL may be drawn from the (given) point A and equal in length to a (given) length BC.

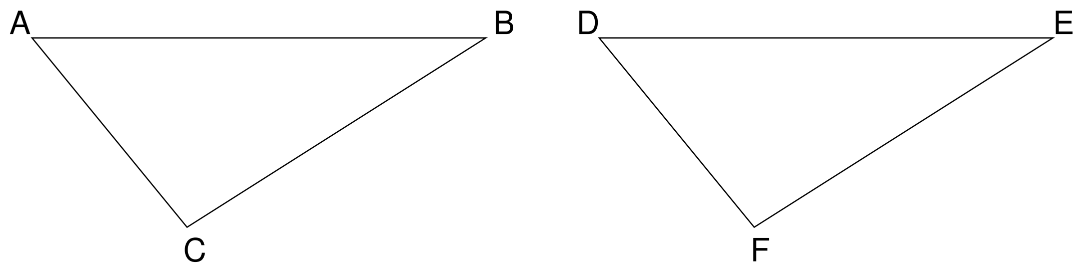

Figure 1

Proposition I.4, by contrast, is a theorem. See Figure 1 below.
ABC and DEF are two triangles, and we are told [given] that:

(the length) AB = (the length) DE

(the length) BC = (the length) EF

(the angle) ABC = (the angle) DEF.

It is then asserted that all the other elements of the first triangle are equal to the corresponding ones in the second. Specifically, the lengths AC and DF are equal, the angles BCA, EFD are equal and the angles CAB, FDE are also equal. Nowadays, we abbreviate this rather long-winded statement by asserting that the triangles ABC and DEF are *congruent*.

I recall still my first encounter with this theorem and its proof. I learnt it from Hall & Stevens' *A School Geometry* (first published in 1903). I found it utterly convincing, and it taught me a fact I had not previously known or even considered. It begins (in the form I learnt it): "Apply the triangle ABC to the triangle DEF." By this is meant that the triangle ABC is imagined to be moved towards the triangle DEF. Euclid then supplies precise directions as to its new position: The point A is to be made to coincide with the point D, and the straight line AB is to overlie the straight line DE. Then, because of the first of the three assumed equalities, the point B must coincide with the point E. But now because of the third equality, the straight line BC will coincide with the straight line EF. Finally, because of the second equality, the points C and F must coincide.

Thus in its new position the first triangle has every one of its points coinciding exactly with those of the second. The result follows.

As I say, I found this proof entirely convincing, and up to a point I still do. However, it has had its critics. Sir Thomas Heath, whose English translation of *The Elements* (Cambridge University Press, 1908) remains standard, notes a number of objections. For example, the French author Jacques Peletier remarked in 1557 that if superposition of lines and figures could be assumed as a method proof, then the whole of geometry would be full of such proofs. He went on to say that if the method was acceptable, then Propositions I.1, I.2, I.3 were unnecessary. In I.2, for example, why not simply pick up the line BC and put it down in such a way that B coincided with A? Heath summed

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up the situation by noting that superposition succeeds as a *practical test*, but is not admissible as a *theoretical proof*. He thought that the Euclidean argument would be better regarded as a reason for assuming I.4 as a postulate. Indeed many mathematicians have done this, among them the eminent German mathematician David Hilbert (who took this course with a minor variation) and Bertrand Russell, who quoted approvingly the direction Hilbert had taken.

It is worth quoting Russell's comments at some length. He wrote (in Chapter 47 of his *Principles of Mathematics*, 1903):

> "The fourth proposition is the first in which Euclid employs the method of superposition a method which, since he will make any détour to avoid it, he evidently dislikes, and rightly, since it has no logical validity, and strikes every intelligent child as a juggle. To speak of motion implies that our triangles are not spatial, but material. For a point of space *is* a position, and can no more change its position than a leopard can change his spots. The motion of a point in space is a phantom directly contradictory to the law of identity: it is the supposition that a given point can be now one point and now another. Hence motion, in the ordinary sense, is only possible to matter, not to space."

Russell goes on in the same vein, but I have probably quoted enough for the reader to grasp the gist of the argument.

I learned of this objection when I was still in high school from a family friend, a mathematics teacher, who summed it up by remarking "You can't pick up triangles!" If we think of the variant of Figure 1 as shown in Figure 2 below, we see an added twist to the objection.

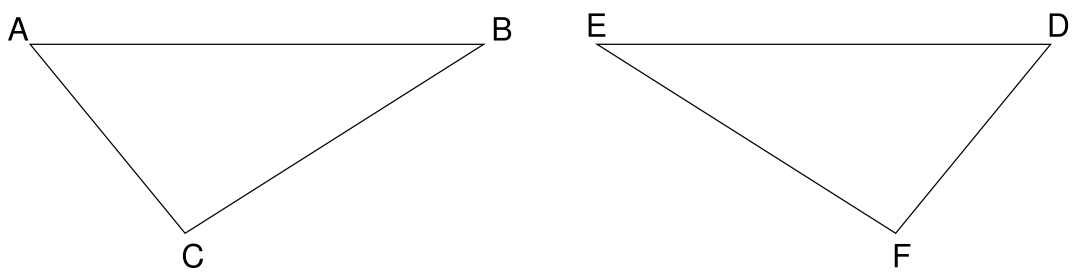

Figure 2

Here if we are to "apply" the triangle ABC to the triangle DEF as specified, it will not be enough merely to slide it about. It also needs to be flipped over. [If we consider, instead of regular triangles, spherical triangles, those whose sides are great circular arcs on a sphere, we see a complication arising from this. For example, take P to be the point on the earth's surface where the latitude and the longitude are both 0^o, and take Q to be the point with latitude 0^o and longitude 90^o ; let N be the North Pole and S the South Pole. Then the hypotheses of I.4 are all applicable to the spherical triangles NPQ and SPQ, but the figures are not congruent. They cannot overlie one another because they curve in opposite directions. The "flipping" fails. They are mirror images of one another, and are sometimes said to be anticongruent. A pair of gloves or of shoes are likewise examples of anticongruence.]

The objectors to Euclid's proof of I.4 make the same point that I made in my dismissal of Dedekind's "proof" on infinite systems, as discussed in my previous column. We are arguing from a physical model of a triangle to establish properties of an abstract ideal, and this is not permissible.

There are two further theorems giving conditions for congruence in Euclid's book. His Proposition I.8 states that in Figure 1 or Figure 2, if AB = DE, BC = EF and CA = FD, then the two triangles are congruent. This also is proved by "applying" one triangle to the other. If we are to avoid the use of application, then we need another postulate. In his revision of the early Euclidean material, Hilbert provides this and so easily proves I.8. However, his extra postulate is almost equivalent to I.8. Euclid's approach applies the triangle ABC to the triangle DEF in such a way that AB coincides with DE. He then uses only ruler and compass constructions, to construct a triangle DEG for which EG = BC and GD = CA, and with G on the same side of DE (i.e. of AB) as is F. Then there are two possibilities: either the point G coincides with F or else it does not. But in I.7, Euclid had shown that only one such point was possible. It is thus impossible for G to be distinct from F. This proves the theorem.

Euclid returned to the topic of conditions ensuring congruence in I.26. This proposition is actually two distinct theorems. The first has the angle ABC equal to the angle DEF, the angle BCA equal to the angle DFE and BC = EF; the second has the same angular equalities, but this time AB = DE. In the first case, the side referred to in each triangle lies between the two angles; in the second case it does not. By assuming that the triangles are not congruent, Euclid produces a contradiction in each of these cases, although the details of the argument differ between them.

In modern terminology, we refer to I.4 as SAS (side-angle-side), to I.8 as SSS (side-side-side), to I.26 (Part 1) as ASA (angle-side-angle) and to I.26 (Part 2) as AAS (angle-angle-side).

Many modern treatments make AAS a simple deduction from ASA, by making use of the theorem that the angles in a triangle together add to two right angles. However, this account depends on the Parallel Postulate, and it is nicer to do without this, as Euclid does.

An interesting question arises as to the equivalence of the different statements on congruence. Euclid deduces all of SSS, ASA and AAS from SAS, although as I have just noted he needed one further resort to an application argument. Can each of these other three be taken as fundamental, and SAS be deduced from these? I expect that they can, but I have not worked out all the details.

Heath remarks that Euclid employed the "application" argument very sparingly, and he speculates (as does Russell) that Euclid was not entirely happy with it himself. (Apart from I.4 and I.8, he made only one further use of it.) This unease may in part explain his failure to use it in one important case. His Postulate 4 (All right angles are equal) can actually be proved by means of an "application" argument. This seems to have been first pointed out by Proclus Diadochus (411-485 CE) in a Commentary on Euclid. Euclid had defined a right angle as one that makes the angles ABD, CBD in Figure 3 to be equal. Proclus assumed that there could be right angles of different sizes and then proceeded to "apply" one version of Figure 3 to another to produce a contradiction.

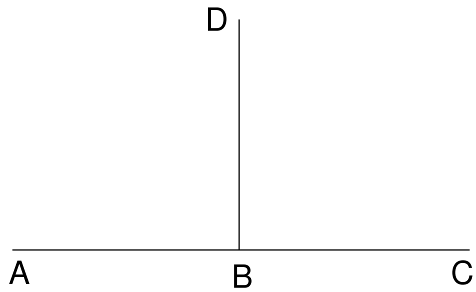

Figure 3

Later, Giovanni Saccheri (1667-1733) also considered the question and produced a variant of Proclus's proof. In his version, he first proved that all straight angles are equal, and then saw a right angle as being half a straight angle. Heath argued that this version is better (principally because it makes its underlying assumptions clearer).

As an aside, Saccheri is an interesting figure. His principal work, published in the year of his death, was a purported proof of Euclid's Parallel Postulate. He began by assuming the postulate to be false, and ultimately arrived at what he believed to be a contradiction. In this he was mistaken; he had unwittingly imported into his argument a step equivalent to what he wanted to prove. (This is very easy to do!) However, he is now credited with the discovery, along the way, of much of what we now call "Non-Euclidean Geometry".

So, current thinking is that the application argument is invalid, and on a strict interpretation, this view must be upheld. However, it is still possible to argue the opposite. Ultimately the point of a proof is to convince a reader or listener of the truth of some statement: a statement that beforehand one might not have believed or considered. My own experience with I.4 (SAS) has left me with a vivid memory of developing conviction of its truth. (So, I must have been an unintelligent child on Russell's reckoning!) After seeing Euclid's proof of I.4 it is not possible to doubt it, at least in the context in which it appears, (although, as my example of the spherical triangles makes clear, there are other contexts in which it does not apply).

I learned many years ago of at least one mathematician who shared this view: Louis Brand. Brand's modern reputation rests very largely on the textbooks he produced, although that is not all he did; he also published several very nice pieces of research mathematics. (I recommended one in my column in *Parabola*,*Volume* **43, 3** (2007), his proof of the Buckingham Π-theorem.) Chapter 3 of his book *Vector and Tensor Analysis* (Wiley, 1947) is concerned with aspects of a subject we call "Differential Geometry".

If we consider a curve winding its way through 3-dimensional space, we can choose some point on it and measure an arclength, s, along it, starting from this point. We can characterize the curve by giving the x, y and z coordinates of each point along the curve as functions of s. However, these functions depend on our choice of the coordinate system, and do not speak directly to the *intrinsic* properties of the curve.

In order to discover these, we define two further functions $\kappa(s)$ and $\tau(s)$, known as the *curvature* and the *torsion* respectively." The fundamental theorem underlying the subject is that if two different curves have the same intrinsic functions $\kappa(s)$ and $\tau(s)$, the same curvature and torsion, then they are congruent. Brand's proof (on p. 97 of his book) begins: "Bring the origins of arc and [the initial orientations of the curves] into coincidence." He might equally well have said "Apply Curve 1 to Curve 2!"