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# Solutions to Problems 1261-1270

**Q1261** A hat contains $N = 2^n$ tickets, $n = 2, 3, 4, \ldots$, each marked with a number from 1 to $N$. (Each ticket has a different number.) In a game, players are asked to draw from the hat two tickets, read them, and replace them. Prize winners are those who draw two numbers whose ratio is 2. The money is refunded if the ratio is $2^k$ for $k = 2, 3, \ldots$. Find the probability of

1. a win;
2. a refund.

**ANS:** (John Barton, Victoria, submitted this solution to the problem.) There are $C_2^N = \dfrac{N!}{2!(N-2)!}$ ways to draw two tickets from $N$ tickets.

1. "Win tickets" are $s$ and $2s$ with $s = 1, \ldots, N/2$. There are $N/2$ such pairs, so

$$P(\text{win}) = \frac{N/2}{C_2^N} = \frac{N}{2}\frac{2}{N(N-1)} = \frac{1}{N-1} = \frac{1}{2^n - 1}.$$

2. "Refund tickets" are $s$ and $2^k s$ where $k = 2, 3, \ldots, n$. Since $2^k s \le N$ we have $s \le 2^{n-k}$. Hence

$$\begin{array}{lcll} k=2 & \to & 2^{n-2} & \text{tickets} \\ k=3 & \to & 2^{n-3} & \text{tickets} \\ \cdots & & \cdots & \\ k=n & \to & 2^0 & \text{ticket.} \end{array}$$

Altogether, there are $2^0 + 2 + \cdots + 2^{n-2} = 2^{n-1} - 1$ pairs. Hence

$$P(\text{refund}) = \frac{2^{n-1}-1}{\frac{N(N-1)}{2}} = \frac{2^n - 2}{2^n(2^n-1)}.$$

**Q1262** Find all functions $f$ satisfying

$$f\left(\frac{x-3}{x+1}\right) + f\left(\frac{x+3}{1-x}\right) = x \quad \text{for all} \quad x \neq \pm 1. \tag{1}$$

**ANS:** Let $t = \frac{x-3}{x+1}$. Then $x = \frac{t+3}{1-t}$ and

$$f(t) + f\left(\frac{t-3}{t+1}\right) = \frac{t+3}{1-t}. \tag{2}$$

Let $t = \frac{x+3}{1-x}$. Then $x = \frac{t-3}{t+1}$ and

$$f\left(\frac{t+3}{1-t}\right) + f(t) = \frac{t-3}{t+1}. \qquad (3)$$

Adding (2) and (3) and using (1) gives

$$2f(t) + t = \frac{8t}{1-t^2},$$

so that

$$f(t) = \frac{4t}{1-t^2} - \frac{t}{2}.$$

**Q1263** Two players $A$ and $B$ play a game with 100 marbles. They take turns to remove the marbles, at least 1 and at most 5 marbles each time. The player who removes the last marbles remaining wins the game.

1. Find a strategy for $A$ to win if he starts the game.
2. Who will lose if initially there are 102 marbles on the table, and both players know the correct strategy?

**ANS:** (John Barton, Victoria, submitted a solution to this problem.) The strategy for a player to win is to leave a multiple of 6 in the pile after his turn. (Why?)

1. For $A$ to win, he should remove 4 marbles in his first turn, leaving 96 marbles on the table, which is a multiple of 6. After that, each time $B$ removes $k$ marbles, $0 < k \leq 5$, $A$ should remove $6 - k$ marbles.
2. If initially there are 102 marbles and both players know the correct strategy then whoever starts the game will lose. (Why?)

**Q1264** David wants to draw 50 points on a square of side length 14cm so that any two points are separated by at least 3cm. Can he do so?

**ANS:** It's impossible! Divide each side of the square into 7 equal intervals to form 49 smaller squares of side length 2cm each. Drawing 50 points results in one small square containing at least 2 points. The largest possible distance of these two points is $\sqrt{2^2 + 2^2} = \sqrt{8} < 3$.

**Q1265** Let $a$, $b$, $c$, and $d$ be positive constants, and $x$ and $y$ be real numbers such that

$$\sin^2 x + \cos^2 y \neq 0 \quad \text{and} \quad \sin^2 y + \cos^2 x \neq 0$$

Prove that

$$\frac{a+b}{c+d} \leq \frac{a\sin^4 x + b\cos^4 y}{c\sin^2 x + d\cos^2 y} + \frac{a\cos^4 x + b\sin^4 y}{c\cos^2 x + d\sin^2 y} \leq \frac{a}{c} + \frac{b}{d}.$$

Find the conditions on $x$ and $y$ such that equalities occur.

**ANS:** Let

$$A = \frac{a\sin^4 x + b\cos^4 y}{c\sin^2 x + d\cos^2 y} + \frac{a\cos^4 x + b\sin^4 y}{c\cos^2 x + d\sin^2 y},$$

$$A_1 = \frac{\sin^4 x}{c\sin^2 x + d\cos^2 y} + \frac{\cos^4 x}{c\cos^2 x + d\sin^2 y}$$

and

$$A_2 = \frac{\cos^4 y}{c\sin^2 x + d\cos^2 y} + \frac{\sin^4 y}{c\cos^2 x + d\sin^2 y}.$$

Then $A = aA_1 + bA_2$.

1. We have

$$A_1 \le \frac{\sin^4 x}{c\sin^2 x} + \frac{\cos^4 x}{c\cos^2 x} = \frac{1}{c} \quad \text{and} \quad A_2 \le \frac{\cos^4 y}{d\cos^2 y} + \frac{\sin^4 y}{d\sin^2 y} = \frac{1}{d},$$

so that

$$A \le \frac{a}{c} + \frac{b}{d}.$$

Equalities occur if

$$\cos^2 x = \cos^2 y = 0 \quad \text{or} \quad \sin^2 x = \sin^2 y = 0.$$

2. Let $B_1 = \sqrt{c\sin^2 x + d\cos^2 y}$ and $B_2 = \sqrt{c\cos^2 x + d\sin^2 y}$. Then by using the inequality

$$(su + tv)^2 \le (s^2 + t^2)(u^2 + v^2) \tag{4}$$

(Cauchy-Schwarz inequality) we obtain

$$\begin{aligned} 1 &= \left(\frac{\sin^2 x}{B_1}B_1 + \frac{\cos^2 x}{B_2}B_2\right)^2 \le \left(\frac{\sin^4 x}{B_1^2} + \frac{\cos^4 x}{B_2^2}\right)\left(B_1^2 + B_2^2\right) \\ &= A_1(c + d), \end{aligned}$$

and

$$\begin{aligned} 1 &= \left(\frac{\cos^2 x}{B_1}B_1 + \frac{\sin^2 x}{B_2}B_2\right)^2 \le \left(\frac{\cos^4 x}{B_1^2} + \frac{\sin^4 x}{B_2^2}\right)\left(B_1^2 + B_2^2\right) \\ &= A_2(c + d), \end{aligned}$$

which imply

$$A_1 \ge \frac{1}{c+d} \quad \text{and} \quad A_2 \ge \frac{1}{c+d}. \tag{5}$$

Therefore,

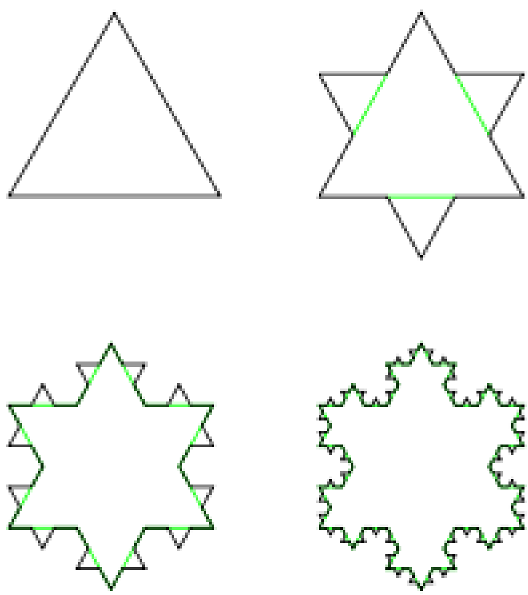
$$A \ge \frac{a+b}{c+d}.$$

The equality in (4) occurs when $s/u = t/v$. Thus, the equalities in (5) occur when

$$\begin{aligned} \frac{\sin^2 x}{c\sin^2 x + d\cos^2 y} &= \frac{\cos^2 x}{c\cos^2 x + d\sin^2 y} \\ &= \frac{\sin^2 x + \cos^2 x}{(c\sin^2 x + d\cos^2 y) + (c\cos^2 x + d\sin^2 y)} \\ &= \frac{1}{c+d}, \end{aligned}$$

implying $\sin^2 x = \cos^2 y$.

**Q1266** Let $C_1$ be an equilateral triangle of side length $a$ cm. We define the curves $C_2$, $C_3$, $C_4$, ... successively from the previous one by trisecting each side and adding an equilateral triangle to the middle section of it, as shown in the figure.



1. Find the perimeter $P_n$ of $C_n$.
2. Find the area $A_n$ of the region bounded by $C_n$.
3. What happens when $n$ gets infinitely large?

**ANS:** Let $s_n$ be the number of sides of $C_n$, and $l_n$ be the length of each side. (Note that $s_1 = 3$ and $l_1 = a$ cm.) To define $C_n$ from $C_{n-1}$, $n = 2, 3, \ldots$, each side is reshaped to 4 sides, so

$$s_n = 4s_{n-1} = 4^2 s_{n-2} = \cdots = 4^{n-1} s_1 = 3 \times 4^{n-1},$$
$$l_n = \frac{1}{3} l_{n-1} = \left(\frac{1}{3}\right)^2 l_{n-2} = \cdots = \left(\frac{1}{3}\right)^{n-1} l_1 = a \left(\frac{1}{3}\right)^{n-1} \text{ cm.}$$

1. The perimeter $P_n$ of $C_n$ is the total length of all its sides:

$$P_n = s_n \times l_n = 3a \left(\frac{4}{3}\right)^{n-1} \text{ cm.}$$

2. Let $a_n$ be the area of the small equilateral triangle added to each side of $C_{n-1}$ to form $C_n$, $n = 2, 3, \ldots$. Then

$$A_n - A_{n-1} = s_{n-1} \times a_n, \quad n = 2, 3, \ldots.$$

Note that

$$\begin{aligned} a_2 &= \frac{1}{9} A_1, \quad a_3 = \frac{1}{9} a_2 = \left(\frac{1}{9}\right)^2 A_1, \quad \text{and} \\ a_n &= \left(\frac{1}{9}\right)^{n-1} A_1, \quad n = 2, 3, \ldots. \end{aligned}$$

Hence

$$A_n - A_{n-1} = \frac{1}{3} \times \left(\frac{4}{9}\right)^{n-2} A_1, \quad n = 2, 3, \ldots.$$

Therefore,

$$\begin{aligned} A_2 - A_1 &= \frac{1}{3} A_1 \\ A_3 - A_2 &= \frac{1}{3} \times \frac{4}{9} A_1 \\ A_4 - A_3 &= \frac{1}{3} \times \left(\frac{4}{9}\right)^2 A_1 \\ &\vdots \\ A_n - A_{n-1} &= \frac{1}{3} \times \left(\frac{4}{9}\right)^{n-2} A_1. \end{aligned}$$

Adding gives

$$\begin{aligned} A_n - A_1 &= \frac{1}{3} \left[1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-2}\right] A_1 \\ &= \frac{3}{5} \left[1 - \left(\frac{4}{9}\right)^{n-1}\right] A_1, \end{aligned}$$

so that

$$A_n = \frac{1}{5}\left[8 - 3\left(\frac{4}{9}\right)^{n-1}\right]\frac{\sqrt{3}a^2}{4}\ \text{cm}^2.$$

3. As $n \to \infty$,

$$P_n \to \infty \quad \text{while} \quad A_n \to \frac{2\sqrt{3}a^2}{5}\ \text{cm}^2,$$

i.e. the perimeter gets infinitely large while the area is about $160\%$ of the original area.

The curve $C_n$ is a Koch curve or Koch snowflake, which is a special case of a geometrical shape called **fractal**.

**Q1267** Find all positive integers $m$ and $n$ satisfying

$$m^3 + n^3 = m^4.$$

**ANS:** (John Barton, Victoria, submitted a solution to this problem.) If $m = n$ then the equation becomes

$$2m^3 = m^4,$$

which has solution $m = 2$. If $m \neq n$ we write the equation as

$$n^3 = m^3(m-1).$$

Hence $m - 1$ is a perfect cube, i.e. $m - 1 = k^3$ for some positive $k$. Then $n = mk$. It is easy to check that all pairs $(m, n)$ with $m = k^3 + 1$ and $n = k(k^3 + 1)$ for $k = 1, 2, 3 \ldots,$ satisfy the equation.

**Q1268** How many integers are of the form

$$a_1 a_2 a_3 \cdots a_{n-1} a_n a_{n-1} \cdots a_3 a_2 a_1$$

, where $0 < a_1 < a_2 < a_3 < \cdots < a_{n-1} < a_n$ and $n \geq 2$? E.g. $272$, $34843$, and $135676531$.

**ANS:** (John Barton, Victoria, submitted a solution to this problem.) A number of the required form has $2k + 1$ digits where $k = 1, 2, \ldots, 8$ (why?). The number is fixed once the first $k + 1$ digits are chosen. There are $\binom{9}{k+1}$ ways to choose the first $k + 1$ digits, so the number of integers of the required form is

$$\begin{aligned}\binom{9}{2} + \binom{9}{3} + \cdots + \binom{9}{9} &= \left[\binom{9}{0} + \binom{9}{1} + \binom{9}{2} + \cdots + \binom{9}{9}\right] \\ &\quad - \left[\binom{9}{0} + \binom{9}{1}\right] \\ &= 2^9 - 10 = 502.\end{aligned}$$

**Q1269** A polyhedron is a solid with planar polygons as its faces. Are there convex polyhedra which have 7 edges?

**ANS:** Assume that there is such a polyhedron. Consider the simplest case when all its faces are triangles. Let $F$ be the number of faces. Since each edge is shared by 2 faces, and a face has 3 edges, there holds $3F = 2 \times 7 = 14$ which is a contradiction. So at least one face is a polygon with more than 3 edges. The simplest convex polyhedron with one face being a quadrilateral is a pyramid, which has 8 edges. Therefore, there are no convex polyhedra having 7 edges.

**Q1270** Prove that if $A$, $B$ and $C$ are three angles of a triangle $\Delta ABC$ then

$$\frac{\sin^2 A + \sin^2 B + \sin^2 C}{\cos^2 A + \cos^2 B + \cos^2 C} \leq 3.$$

When does the equality occur?

**ANS:** Since

$$\begin{aligned} \frac{\sin^2 A + \sin^2 B + \sin^2 C}{\cos^2 A + \cos^2 B + \cos^2 C} &= \frac{1 - \cos^2 A + 1 - \cos^2 B + 1 - \cos^2 C}{\cos^2 A + \cos^2 B + \cos^2 C} \\ &= \frac{3}{\cos^2 A + \cos^2 B + \cos^2 C} - 1, \end{aligned}$$

the required inequality is equivalent to

$$\cos^2 A + \cos^2 B + \cos^2 C \geq \frac{3}{4}. \tag{6}$$

Since $A + B + C = \pi$ we have

$$\begin{aligned} \cos^2 A + \cos^2 B + \cos^2 C &= \cos^2 A + \frac{1 + \cos 2B}{2} + \frac{1 + \cos 2C}{2} \\ &= \cos^2 A + \cos(B + C)\cos(B - C) + 1 \\ &= \cos^2 A - \cos(B - C)\cos A + 1 \\ &= \left(\cos A - \frac{1}{2}\cos(B - C)\right)^2 - \frac{1}{4}\cos^2(B - C) + 1 \\ &\geq \frac{1}{4}\sin^2(B - C) + \frac{3}{4} \\ &\geq \frac{3}{4}. \end{aligned}$$

Equality occurs when $\cos A = \frac{1}{2}\cos(B - C)$ and $\sin(B - C) = 0$, implying $A = B = C = \pi/3$, i.e. $\Delta ABC$ is an equilateral triangle.