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Discovering Mathematics: A Case Study

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As a broad generalisation we might say that research in mathematics consists of two parts: finding out what is true, and proving that it is true. Mathematical articles (even those written for school students) traditionally concentrate on the second of these, covering up the investigator's tracks and offering only the final, beautifully clear path to the proved theorem. Readers may find it of interest to see where the ideas came from in a problem I tackled recently.

In 2008 I received an email asking a question about the value of an infinite *generalised continued fraction*

$$\frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \cdots}}}. \tag{$*$}$$

The dots indicate that the numbers a_k and b_k go on for ever. What does such an expression mean, and how do we evaluate it? We truncate the expression after each a_k to give the (finite) fractions

$$\frac{b_1}{a_1} \quad \text{and} \quad \frac{b_1}{a_1 + \frac{b_2}{a_2}} \quad \text{and} \quad \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3}}}$$

and so on, which are called the *convergents* of the continued fraction. We then compute each of these and see what happens to our results as we move along the sequence of convergents. If they get closer and closer to some specific value then we say that the infinite continued fraction *converges* to this value, if not, then the continued fraction cannot be assigned any sensible value and we say that it *diverges*.

One final introductory comment: in order to save space we usually write the continued fraction $(*)$ as

$$\frac{b_1}{a_1+}\frac{b_2}{a_2+}\frac{b_3}{a_3+}$$

(with the plus signs in the denominator, not between the fractions). Don't forget that in this notation each fraction line is taken to include *all* of the expression to its right.

The most interesting continued fractions (at least for number theorists) are those in which all the values a_k and b_k are positive integers: we shall assume this to be the case throughout the present article. An important question is whether a given continued

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fraction converges to a rational or an irrational value. To get a feel for all of these ideas you might like to calculate the first six or seven convergents of the continued fraction

$$\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots,$$

make a guess at the value of the continued fraction, and decide whether it is rational or irrational. **Hint** If you can't see any pattern in the convergents, try looking at their squares.

The question posed by my correspondent was the following: if $a_1 < a_2 < a_3 < \cdots$ and $b_1 < b_2 < b_3 < \cdots$, is it true that the continued fraction must necessarily converge to an irrational value? With an open–ended question like this one we might as well start by looking at a few examples. Given the conditions on the a_k and b_k, the simplest possible example is

$$\frac{1}{1+}\,\frac{2}{2+}\,\frac{3}{3+}\,\frac{4}{4+}\cdots.$$

Now as it happens this was a fraction I was already familiar with: it converges to the value $1/(e-1)$, which is irrational, and therefore does not answer the question. (It shows that the kind of expression we are considering *may* be irrational, but not that it *must* be irrational.)

So, needing a slightly more complicated example, I thought of increasing each numerator by 1 and investigating the continued fraction

$$\frac{2}{1+}\,\frac{3}{2+}\,\frac{4}{3+}\,\frac{5}{4+}\cdots.$$

How can we go about evaluating something like this? As the value of the continued fraction is the limit of its convergents, let's calculate a few convergents – using computer assistance of course!! This gives the fractions

$$\frac{2}{1},\ \frac{4}{5},\ \frac{20}{19},\ \frac{100}{101},\ \frac{620}{619},\ \frac{4420}{4421},\ \frac{35900}{35899},\ \frac{326980}{326981}.$$

We have clearly come up with a lucky guess as the answer is staring us in the face: each numerator is 1 more or 1 less than the corresponding denominator, and so as we proceed along the sequence the fractions will become closer and closer to 1. That is, we guess that

$$\frac{2}{1+}\,\frac{3}{2+}\,\frac{4}{3+}\,\frac{5}{4+}\cdots = 1.$$

Now we would like to prove that this guess is really correct and not just the result of an amazing coincidence. (Stranger things have happened in mathematics!) This is not something I wish to do in the present article, but I'll just report that I did complete the proof by showing that the kth convergent has denominator

$$(k+1)! - k! + (k-1)! - (k-2)! + \cdots \pm 1!\,.$$

As we have shown that a continued fraction of the type under consideration *may* be rational, the original question is answered in the negative. But we often find that the solution to one question, if it's a good question, will suggest further problems. In this case it made me wonder what would have happened if I had increased the numerators by more than 1. That is, I want to investigate the expression

$$f(s) = \frac{1+s}{1+}\,\frac{2+s}{2+}\,\frac{3+s}{3+}\,\frac{4+s}{4+}\cdots,$$

where s is a positive integer. Once again we start by calculating a few convergents. If $s = 2$ we obtain

$$\frac{3}{1}, \frac{1}{1}, \frac{33}{23}, \frac{21}{16}, \frac{119}{89}, \frac{777}{583}, \frac{64029}{48019}, \frac{49161}{36871},$$

and this time there is no clear pattern to the numerators and denominators. This may not be a problem as we are not really interested in the numerators and denominators separately but only in their quotients. And calculating a few of these as decimals,

$$\begin{aligned}&3.0,\ 1.0,\ 1.434782609,\ 1.312500000,\ 1.337078652,\\ &\quad 1.332761578,\ 1.333409692,\ 1.333324293,\ 1.333334294,\\ &\quad 1.333333241,\ 1.333333341,\ 1.333333333,\ 1.333333333,\ldots,\end{aligned}$$

makes our next guess pretty obvious:

$$f(2) = \frac{3}{1+}\,\frac{4}{2+}\,\frac{5}{3+}\,\frac{6}{4+}\cdots = 1.333333333\cdots = \frac{4}{3}.$$

If we follow through the same ideas for $s = 3, 4, 5, \ldots$ it appears that the continued fractions converge to the values

$$\begin{aligned} f(3) &= 1.615384615\\ f(4) &= 1.863013699\\ f(5) &= 2.085828343\\ f(6) &= 2.289804986\\ f(7) &= 2.478914782. \end{aligned}$$

In the first of these results there is just the possibility that we can see the beginning of a repeating decimal, which can easily be converted into a rational number: we have the very tentative guess that

$$f(3) = 1.615384615384615384\cdots = 1 + \frac{615384}{999999} = \frac{21}{13}.$$

There does not seem to be much we can say about the following numbers. However, if we are willing to risk the guess that they too are rational (and why not? the floor won't cave in beneath us if we are wrong), there is a well-established procedure to test this. We expand each number in a *simple* continued fraction, that is, one where every numerator is 1. Doing so by computer we find, for example, that

$$f(4) = 1.863013699 = 1 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{507356+}\cdots.$$

Now it is a known fact about simple continued fractions that large partial quotients such as 507356 are exceedingly rare; we therefore surmise that this term is due to inaccuracies in our calculations, that it should not really be there, and that we should actually have a *finite* continued fraction

$$f(4) = 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{6+}\frac{1}{3+}\frac{1}{3} = \frac{136}{73}.$$

If we continue for further values of s we find, as far as we could be bothered calculating, that there is *always* a very large term in the continued fraction! So we suspect that all the expressions $f(s)$ are rational, and in particular that $f(s)$ is

$$\frac{1}{1}, \quad \frac{4}{3}, \quad \frac{21}{13}, \quad \frac{136}{73}, \quad \frac{1045}{501}, \quad \frac{9276}{4051}, \quad \frac{93289}{37633}$$

for $s = 1, 2, 3, 4, 5, 6, 7$.

Don't forget that all of this still awaits proof, and it might be wrong! Everything we have stated so far is merely conjecture (a polite term: as mathematicians don't seem to like the word 'guess'). However the evidence we have is fairly strong: we would probably now be thinking of how we might prove all this, and maybe even find a formula for the fractions $f(s) = u(s)/v(s)$.

Another good way of finding things out in mathematics is to see whether perhaps someone else may have found them out already. 'Unless we take the trouble to learn from our predecessors, mathematics might as well be written on water.'[2]

A superb resource for identifying sequences of integers is the On-line Encyclopedia of Integer Sequences. [3] If we go to this site and enter the first few terms of a sequence, the encyclopedia will rapidly return any potential matches. Doing this with the numerators $u(s) = 1, 4, 21, 136, 1045, 9276, 93289$ of the above fractions gave only one result, and so it seems quite likely to be what we are looking for; doing the same thing with the denominators also gave only one result, and by this stage we should be almost completely convinced that we are on the right track.

'Almost completely': after all, we have only checked the first seven terms of each sequence against the OEIS; it could conceivably be that they differ at a later stage. However, a bit more rummaging in the OEIS gives some formulae which, when translated into our notation, can be written

$$u(s) = su(s-1) + sv(s-1) \text{ and } v(s) = u(s-1) + sv(s-1) \,.$$

We now have another conjecture, and although there is much work still to be done, we can try to prove that these formulae always apply and that they give the values of $f(s)$.

Another benefit of looking through the OEIS is that it gives copious references. None of those for our two sequences has any relation to continued fractions, so it appears at least possible that we have discovered something which nobody else has ever noticed!

[2] Thomas Körner, *The Pleasures of Counting*, Cambridge University Press (1996).

[3] http://oeis.org

I don't want to talk about the actual proof, except to say that it was successful, so this is the end of the story. Perhaps it has illustrated a side of mathematics that you may have seen little of before. I hope you will agree that mathematics is about far more than memorising formulae; that you have to be prepared to make guesses and see if they stand up to examination; and that, apart from anything else, working through an investigation is simply much more fun than merely contemplating the final result!

Answer to the above question: the continued fraction converges,

$$\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\cdots = \frac{1}{\sqrt{3}},$$

which is irrational.