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History of Mathematics: Diagrams

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A 2007 book tells a very interesting story. It is called *The Archimedes Codex* and it recounts how a lost work by that great Greek mathematician recently came to light. The authors are Reviel Netz, a mathematical historian, and William Noel, the curator of the museum that holds it. My purpose here is not to repeat what is said in their book, but rather to pick up on a point Netz makes in the course of his contribution to it. Modern mathematics, he says, is a science of equations; ancient science was a science of diagrams. He continues: 'In modern science, diagrams serve as a kind of illustration; they are there to make the experience of learning science somewhat less traumatic for the student'. But they are not part of the logic of the argument itself. In modern science it is considered crucial to make sure that no information depends on the diagram to make sure the logic of the proof works in its full, most general form, we must rely on the language alone and never on the diagram. In my undergraduate days, we surely took this moral to heart. We sang with great gusto, to the tune *Men of Harlech*, but possibly also somewhat tongue-in-cheek, a ditty called *The Pure Mathematicians Anthem*. One verse ran:

"With an energy fanatic

Guard each little mathematic

From a treatment di'grammatic

Keep Math'matics pure!"

This attitude was very much that of a very great mathematician of the 18^{th} century, Joseph-Louis Lagrange (1736-1813). Introducing his 1788 work *Mécanique Analytique*, he wrote a passage that in English translation reads:

> No figures will be found in this work. The methods I present require neither constructions nor geometrical or mechanical arguments, but solely algebraic operations subject to a regular and uniform procedure. Those who appreciate mathematical analysis will see with pleasure mechanics becoming a new branch of it and hence, will recognize that I have enlarged its domain.

Things were quite different, however, for the ancient Greeks, as Netz is at pains to point out.

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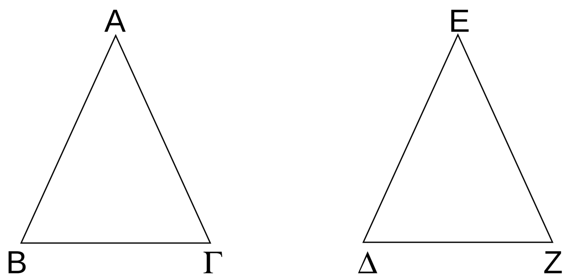

> A Greek proposition makes its claim in terms of [its] diagram. This is the only place where the points and lines of the proposition are provided with meaning. This is done through alphabetic labels, in exactly the way we do today; in fact, in this we follow a Greek invention.

It seems that Netz was the first to point this out so explicitly. In fact he says he is known as the guy who made that observation on Greek diagrams. However, once the point is made, it is clear that it is correct. The first three propositions of Euclid's *Elements* are all what today we would call Constructions; they are sets of instructions to be followed in the accurate drawing of diagrams. Once we are assured that a diagram may be constructed, we are then able to use it in order to advance the argument of the other type of Proposition, the Theorem.

Netz illustrates his point by reference to an argument Archimedes uses in discussing two cones labelled $AB\Gamma$ and ΔEZ. The six letters in fact label the vertices of triangles made by cutting through the cones, a matter, which Netz says, 'we may easily guess'.

But he continues:

> But how are we to know the individual distribution of the letters? In each cone, two letters must stand on the base, and one on the top but which is which? This is what makes this observation [the crucial role of the diagram] so difficult to make: because visual information is so powerful, the moment we are in front of a diagram we immediately read off the information and establish that $B\Gamma$, ΔZ are bases, A, E are tops; and we even fail to notice that *the text said no such thing*. In fact, this is the general rule throughout Greek mathematics; the identity of objects is not established by the words but by the diagrams. The diagrams are there not as some kind of illustration, so as to make the reading experience more pleasant; the diagrams are there to provide us with the most basic information. They tell us the Who's Who of the proposition: which letter stands next to which object. Ancient diagrams are not illustrative, they are informative; they constitute part of the logic of the proposition.

One of the more surprising aspects of the diagram's role, however, is the fact that the diagram need not be accurate. Netz shows that Archimedes displayed straight lines as curved in order to make them more visible to the reader. However, the constructions given by Euclid and accepted by other ancient Greek mathematicians, including Archimedes, always, in principle, result in accurate diagrams. It's as if, once you know that something *can* be done, we do not always see the need actually to do it! Today, we think rather differently, and this is one reason why we find it hard to go back to the earlier style of thought. The modern approach has been well summed up by the contemporary philosopher and mathematician Kathryn Mann:

> Everybody knows the proper place of diagrams in mathematics. When a mathematician explores new ideas or explains concepts to others, diagrams are useful, even essential. When she instead wishes to formally prove a theorem, diagrams must be swept to the side.

Professor Mann credits the great German mathematician David Hilbert (1862-1943) with the most obvious change in attitude. It was Hilbert's sentence-based *Grundlagen der Geometrie* that replaced Euclid's diagrammatic *Elements* as the foundation of geometry. The *Grundlagen der Geometrie [Foundations of Geometry]* is widely regarded as a worthy (the most worthy) successor to Euclid's enterprise. It puts Euclid's work into a modern context and replaces the list of axioms that he used by a more complete set, filling in gaps left in the earlier work. I briefly referred to this work in an earlier column (*Parabola* Volume 44, Number 2), which discussed a controversial proof in the *Elements*.

A recent historian of mathematics, Henk Bos, says that the effect of Hilbert's work was to cut the bond with reality. One of Hilbert's most memorable sayings was that the terms *point*, *line* and *plane* used in Euclidean geometry could be replaced by *table*, *chair* and *beer mug* and the logical structure would still be the same. I leave readers to contemplate how this would alter the diagrams! (The intersection of two beer mugs is a chair!) But, perhaps remarkably, Hilbert *did* use diagrams. His *Grundlagen*, despite his quip, has quite a lot of them. In fact, he held diagrams in high regard. Writing in 1902, a few years after his *Grundlagen* appeared, he said,

> Arithmetic symbols are written diagrams and geometrical figures are graphical formulas.

[He was actually echoing (perhaps unknowingly) an earlier statement by Sophie Germain (1776-1831):

> Algebra is but written geometry and geometry is but figured [i.e. diagrammatic] algebra.

But her statement, if you examine it closely, has a somewhat different emphasis.

Hilbert was later to expand on his rather cryptic statement. In the preface to his book *Geometry and the Imagination*, he says

> In mathematics ... we find two tendencies present. On the one hand, the tendency towards *abstraction* seeks to crystallize the *logical relations* inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward *intuitive understanding* fosters a more immediate grasp of the objects one studies, a live rapport with them, so to speak, which stresses the concrete

> meaning of their relations. [Many mathematical theories] make extensive use of abstract reasoning and symbolic calculation in the sense of algebra. Notwithstanding this, it is as true today as it ever was that *intuitive* understanding plays a major role in geometry. And such concrete intuition is of great value not only for the research worker, but also for anyone who wishes to study and appreciate the results of research in geometry.

The position outlined here is the same as that summarized by Mann and by Netz (in his characterization of the modern outlook). The logical structure is paramount; the diagrams are secondary, but extremely useful in guiding intuition and, as Netz has it, 'mak[ing] the experience of learning science somewhat less traumatic for the student'. The pitfalls involved in relying entirely on diagrams are well illustrated by the more recent author Howard Eves in his book *A Survey of Geometry*. He discusses the very first Proposition in the *Elements*. This is a construction and it gives a recipe for constructing an equilateral triangle on a given base *AB*. This runs as follows:

> With center *A* and radius *AB* draw a circle.
>
> With center *B* and radius *BA* draw another circle.
>
> Let *C* be one of the points where the two circles intersect.
>
> Then *ABC* is the required triangle.

Readers will readily appreciate that this recipe works (try it yourself if you're not already convinced), so what is wrong with it? Well, although it is visually obvious that the two circles intersect (in two places), there is no axiom or postulate in Euclid's scheme that makes this the case. As Eves writes:

> there is nothing in Euclid's first principles which explicitly guarantees that the two circles shall intersect in a point *C*, and that they will not, somehow or another, slip through each other with no common point. The existence of this point, then, must be either postulated or proved, and it can be shown that Euclid's postulates are insufficient to permit the latter. Only by the introduction of some additional assumption can the existence of the point *C* be established. Therefore the proposition does not follow from Euclid's first principles, and the proof of the proposition is invalid.

The point is a subtle one, and it takes some thought to appreciate its full force. We are so seduced by the obviousness of the existence of the point(s) *C* that the fact that this requires justification eludes us. Seeing is believing. At first, the reader may think that it is simply a piece of pedantry to make much of this matter, but if we reflect that Euclid's purpose was to develop the *logical structure* behind the visually obvious, then we can see that Eves is right to make this objection to this gap in his logic. The issue is the converse of the one raised by Netz who said:

> because visual information is so powerful, the moment we are in front of a diagram we immediately read off the information and establish that B, Z are bases, A, E are tops; and we even fail to notice that *the text said no such thing*.

We might say:

> because visual information is so powerful, the moment we are in front of a diagram we immediately read off the information and establish that C exists and we even fail to notice that *the axioms and postulates guarantee no such thing*.