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Problems 1291–1300

Q1291 Show that there do not exist three primes x, y and z satisfying

$$x^2 + y^3 = z^4.$$

Q1292 Prove that for all a, b, c, and d satisfying $0 \le a, b, c, d \le 1$, there holds

$$\frac{a}{b+c+d+1} + \frac{b}{c+d+a+1} + \frac{c}{d+a+b+1} + \frac{d}{a+b+c+1} + (1-a)(1-b)(1-c)(1-d) \le 1.$$

Q1293 Suppose that $u = \cot(\pi/8)$ and $v = \text{cosec}(\pi/8)$. Prove that u satisfies a quadratic and v a quartic equation with integral coefficients and with leading coefficients 1.

Q1294 Let a and b be two sides of a triangle, and α and β be two angles opposite these sides, respectively. Prove that

$$\frac{a+b}{a-b} = \frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}.$$

Q1295 Assume that the following information about a triangle is known: the radius R of the circumscribed circle, the length c of one side, and the ratio a/b of the lengths of the other two sides. Determine all three sides and angles of this triangle.

Q1296 Let a, b and c be the sides, and m_a, m_b and m_c be the medians of a triangle ABC. Prove that

$$m_a^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2), \quad m_b^2 = \frac{1}{4}(2c^2 + 2a^2 - b^2), \quad m_c^2 = \frac{1}{4}(2a^2 + 2b^2 - c^2).$$

Q1297 (Suggested by Dr. Panagiote Ligouras, Leonardo da Vinci High School, Noci, Bari, Italy)

Let a, b and c be the sides, and m_a, m_b and m_c be the medians of a triangle ABC. Prove or disprove that

$$27(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \le 64(m_a^4 + m_b^4 + m_c^4)(m_a^2 + m_b^2 + m_c^2).$$

Q1298 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(a+b) + f(b+c) + f(c+a) \ge 3f(a+2b+3c) \quad \text{for all } a, b, c \in \mathbb{R}.$$

Q1299 Let f be a function satisfying each of the following

1. For all real numbers x and y, there holds

$$f(x+y) + f(x-y) = 2f(x)f(y).$$

2. There exists a real number a such that $f(a) = -1$.

Prove that f is periodic.

Q1300 Find all polynomials $p(x)$ satisfying

$$(x-16)p(2x) = 16(x-1)p(x) \quad \text{for all } x \in \mathbb{R}.$$