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Solutions to Problems 1281-1290

Q1281 Prove that for any real numbers a and b there holds

$$\frac{1+|a|}{1+|b|} \leq 1+|a-b|.$$

ANS: If $|a| \leq |b|$ then the left-hand side is less than or equal to 1, whereas the right-hand side is greater than or equal to 1. So the inequality is true. Now if $|a| > |b|$, then

$$\frac{1+|a|}{1+|b|} = 1 + \frac{|a|-|b|}{1+|b|} \leq 1 + \frac{|a-b|}{1+|b|} \leq 1+|a-b|.$$

Here we use the triangular inequality

$$||a|-|b|| \leq |a-b| \quad \text{for any } a, b \in \mathbb{R},$$

which can be proved by squaring both sides and noting that $ab \leq |a||b|$ or, equivalently, $-2|a||b| \leq -2ab$.

Q1282 Let a and b be two real numbers satisfying $0 \leq a, b \leq 1/2$ and $a+b > 0$. Show that

$$\frac{ab}{(a+b)^2} \leq \frac{(1-a)(1-b)}{(2-a-b)^2}.$$

ANS: Showing this inequality is true is equivalent to showing

$$ab(2-a-b)^2 \leq (1-a)(1-b)(a+b)^2.$$

Some calculation reveals that this is equivalent to showing

$$a^3+b^3-a^2b-ab^2 \leq (a-b)^2.$$

It is easy to prove this inequality. In fact, since $a+b \leq 1$ there holds

$$LHS = (a+b)(a-b)^2 \leq (a-b)^2 = RHS.$$

Q1283 Generalise the result of **Q1282** to the case of three numbers a, b, and c.

ANS: For all real numbers a, b, and c satisfying $0 \leq a, b, c \leq 1/2$ and $a+b+c > 0$ there holds

$$\frac{abc}{(a+b+c)^3} \leq \frac{(1-a)(1-b)(1-c)}{(3-a-b-c)^3}. \tag{0.1}$$

To prove this inequality we first prove that if $0 \le a, b, c, d \le 1/2$ and $a+b+c+d > 0$ then
$$\frac{abcd}{(a+b+c+d)^4} \le \frac{(1-a)(1-b)(1-c)(1-d)}{(4-a-b-c-d)^4}. \tag{0.2}$$
This inequality can be proved by using the result of **Q1282** successively to show
$$\sqrt{\frac{ab}{(1-a)(1-b)}} \le \frac{a+b}{2-a-b} \quad \text{and} \quad \sqrt{\frac{cd}{(1-c)(1-d)}} \le \frac{c+d}{2-c-d},$$
implying
$$\begin{aligned}\sqrt{\frac{abcd}{(1-a)(1-b)(1-c)(1-d)}} &\le \frac{(a+b)(c+d)}{(2-a-b)(2-c-d)}\\ &= \frac{\left(\frac{a+b}{2}\right)\left(\frac{c+d}{2}\right)}{\left(1-\frac{a+b}{2}\right)\left(1-\frac{c+d}{2}\right)}.\end{aligned}$$
By using the result of **Q1282** with two numbers $(a+b)/2$ and $(c+d)/2$ we deduce
$$\begin{aligned}\sqrt{\frac{abcd}{(1-a)(1-b)(1-c)(1-d)}} &\le \frac{\left(\frac{a+b+c+d}{2}\right)^2}{\left(2-\frac{a+b}{2}-\frac{c+d}{2}\right)^2}\\ &= \left(\frac{a+b+c+d}{4-a-b-c-d}\right)^2,\end{aligned}$$
implying (0.2). We now use (0.2) for four numbers a, b, c, and $(a+b+c)/3$ to show
$$\frac{abc\frac{a+b+c}{3}}{(1-a)(1-b)(1-c)\left(1-\frac{a+b+c}{3}\right)} \le \left(\frac{a+b+c+\frac{a+b+c}{3}}{4-a-b-c-\frac{a+b+c}{3}}\right)^4,$$
implying
$$\frac{abc}{(1-a)(1-b)(1-c)} \times \frac{a+b+c}{3-a-b-c} \le \left(\frac{a+b+c}{3-a-b-c}\right)^4.$$
By cancelling the common term on both sides we obtain (0.1).

Q1284 Let A_1, B_1, and C_1 be three points on the sides BC, CA, and AB (respectively) of a triangle ABC. Show that
$$\frac{AC_1}{C_1B}\,\frac{BA_1}{A_1C}\,\frac{CB_1}{B_1A} = \frac{\sin\angle ACC_1}{\sin\angle C_1CB}\,\frac{\sin\angle BAA_1}{\sin\angle A_1AC}\,\frac{\sin\angle CBB_1}{\sin\angle B_1BA}.$$

ANS: On the one hand,
$$\frac{\text{area}(ACC_1)}{\text{area}(BCC_1)} = \frac{AC_1}{BC_1}.$$

On the other hand,
$$\frac{\text{area}(ACC_1)}{\text{area}(BCC_1)} = \frac{AC \sin \angle ACC_1}{BC \sin \angle BCC_1}.$$
Hence
$$\frac{AC_1}{BC_1} = \frac{AC \sin \angle ACC_1}{BC \sin \angle BCC_1}.$$
Similarly,
$$\frac{BA_1}{CA_1} = \frac{AB \sin \angle BAA_1}{AC \sin \angle A_1AC} \quad \text{and} \quad \frac{CB_1}{AB_1} = \frac{BC \sin \angle CBB_1}{AB \sin \angle B_1BA}.$$
The result now follows from multiplying three expressions.

Q1285 Let ABC be a triangle such that sides AB and AC are fixed, but the angle $\angle BAC$ may vary. From the exterior of ABC, construct 3 squares $ABDE$, $ACGF$, and $BCHK$. Find the angle $\angle BAC$ such that the area of the hexagon $DEFGHK$ is maximum.

ANS: Let $a = BC$, $b = AC$, $c = AB$, $\alpha = \angle BAC$, and $\beta = \angle ABC$. Then
$$\text{area}(ABC) = \frac{1}{2}bc \sin \alpha \quad \text{and} \quad \text{area}(AEF) = \frac{1}{2}bc \sin \alpha,$$
noting that $\angle EAF = \pi - \alpha$. Since $\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}$ we also have
$$\text{area}(BDK) = \frac{1}{2}ac \sin \beta = \frac{1}{2}bc \sin \alpha.$$
Similarly
$$\text{area}(CHG) = \frac{1}{2}bc \sin \alpha.$$
Therefore,
$$\begin{aligned}
S &= \text{area}(DEFGHK) \\
&= \text{area}(AEDB) + \text{area}(ACGF) + \text{area}(BCHK) \\
&\quad + \text{area}(ABC) + \text{area}(AEF) + \text{area}(BDK) + \text{area}(CHG) \\
&= a^2 + b^2 + c^2 + 2bc \sin \alpha \\
&= 2(b^2 + c^2) + 2bc(\sin \alpha - \cos \alpha) \\
&= 2(b^2 + c^2) + 2bc \sin(\alpha - \frac{\pi}{4}).
\end{aligned}$$
Hence $S = S_{\max}$ when $\sin(\alpha - \frac{\pi}{4}) = 1$, which is the case when $\alpha = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4} = 135°$.

Q1286 From the exterior of a triangle ABC, draw 3 equilateral triangles ABX, BCY, and CAZ, whose corresponding centroids are M, N, and K. Show that MNK is an equilateral triangle.

ANS: Let $a = BC$, $b = AC$, $c = AB$, $\alpha = \angle BAC$, $\beta = \angle ABC$, and $\gamma = \angle ACB$. Then $AM = c/\sqrt{3}$, $AK = b/\sqrt{3}$, and $\angle MAK = \alpha + 60^\circ$ so that

$$\begin{aligned} MK^2 &= AM^2 + AK^2 - 2AM \times AK \times \cos(\angle MAK) \\ &= \frac{1}{3}\left(c^2 + b^2 - 2bc\cos(\alpha + 60^\circ)\right) \\ &= \frac{1}{3}(c^2 + b^2 - bc\cos\alpha + \sqrt{3}bc\sin\alpha). \end{aligned}$$

On the other hand,

$$S = \text{area}(ABC) = \frac{1}{2}bc\sin\alpha \quad \text{and} \quad b^2 + c^2 - a^2 = 2bc\cos\alpha.$$

Hence

$$MK^2 = \frac{1}{6}(a^2 + b^2 + c^2 + 4\sqrt{3}S).$$

Similarly,

$$MN^2 = NK^2 = \frac{1}{6}(a^2 + b^2 + c^2 + 4\sqrt{3}S).$$

Therefore, MNK is an equilateral triangle.

Q1287 Let A and B be two points on the parabola $y = x^2$. Assume that $AB = 2$. Find the positions of A and B such that the area of the region formed by AB and the parabola is maximum.

ANS: Let the coordinates of A and B be $A(x_1, x_1^2)$ and $B(x_2, x_2^2)$. Without loss of generality we can assume that $x_1 < x_2$. If $A_1(x_1, 0)$ and $B_1(x_2, 0)$ then

$$\begin{aligned} S &= \text{area}(ABB_1A_1) - \int_{x_1}^{x_2} x^2\, dx \\ &= \frac{1}{2}(x_2 - x_1)(x_1^2 + x_2^2) - \frac{1}{3}\, x^3\Big|_{x_1}^{x_2} \\ &= \frac{1}{6}(x_2 - x_1)^3. \end{aligned}$$

Hence $S = S_{\max}$ when $(x_2 - x_1)^2$ attains the maximum value. Since

$$4 = AB^2 = (x_2 - x_1)^2 + (x_2^2 - x_1^2)^2,$$

there holds

$$(x_2 - x_1)^2 = \frac{4}{1 + (x_1 + x_2)^2}.$$

Therefore, $(x_2 - x_1)^2$ attains the maximum value when $x_1 + x_2 = 0$, i.e. $x_1 = -x_2$. It follows that $x_1 = -1$ and $x_2 = 1$.

Q1288 Show that there are no integers x and y satisfying

$$x^2 - 2y^2 = 5. \tag{0.3}$$

ANS: Assume that there exist two integers x and y satisfying (0.3). We can assume that $x, y \geq 0$. Equation (0.3) can be rewritten as

$$x^2 - 1 = 2y^2 + 4,$$

or

$$(x-1)(x+1) = 2(y^2+2).$$

This implies that $(x-1)(x+1)$ is even, and hence $x-1$ and $x+1$ are also even. Thus

$$x - 1 = 2n \quad \text{and} \quad x + 1 = 2n + 2 \quad \text{for some integer } n.$$

Then $y^2 + 2 = 2n(n+1)$, i.e. y is even, namely $y = 2k$ for some integer k. It follows that

$$n(n+1) = 2k^2 + 1,$$

which is a contradiction because the LHS is even whereas the RHS is odd.

Q1289 Two chess teams play against each other in a competition. The teams have different numbers of players, and one team has an odd number of players. Each player has to play one game with each player of the other team. The total games played are 4 times the total players of both teams. How many players has each team?

ANS: Let x and y be the numbers of players, where x is odd. Then

$$xy = 4(x+y).$$

It follows that

$$y = \frac{4x}{x-4} = 4 + \frac{16}{x-4}.$$

Since x and y are integers, $x - 4$ must be a divisor of 16, namely,

$$x - 4 = \pm 1, \pm 2, \pm 4, \pm 8, \text{ or } \pm 16.$$

Because x is odd, the only solutions are $x = 3$ or $x = 5$. But if $x = 3$ then $y = -12 < 0$. Therefore, $x = 5$ and $y = 20$.

Q1290 Prove that from any 4 real numbers we can choose x and y satisfying

$$0 \leq \frac{x-y}{1+xy} \leq 1.$$

ANS: The expression $\frac{x-y}{1+xy}$ reminds us of

$$\frac{\tan\beta - \tan\alpha}{1 + \tan\alpha\tan\beta} = \tan(\beta - \alpha). \tag{0.4}$$

Now, let $a \leq b \leq c \leq d$ be any 4 real numbers. There exist $\alpha, \beta, \gamma, \delta \in (-\pi/2, \pi/2)$ such that

$$a = \tan\alpha, \quad b = \tan\beta, \quad c = \tan\gamma, \quad d = \tan\delta.$$

Note that

$$\alpha \leq \beta \leq \gamma \leq \delta \leq \alpha + \pi.$$

This means that β, γ and δ divide the interval $[\alpha, \alpha+\pi]$ into 4 subintervals, namely, $[\alpha, \beta]$, $[\beta, \gamma]$, $[\gamma, \delta]$ and $[\delta, \alpha+\pi]$. At least one of these subintervals has length less than or equal to $\pi/4$.

Assume that $[\alpha, \beta]$ is such a subinterval. Then $0 \leq \beta - \alpha \leq \pi/4$, implying $0 \leq \tan(\beta - \alpha) \leq 1$. It follows from (0.4) that

$$0 \leq \frac{b-a}{1+ab} \leq 1.$$

A similar argument holds if $[\beta, \gamma]$ or $[\gamma, \delta]$ has length less than or equal to $\pi/4$. Finally, if $[\delta, \alpha+\pi]$ has length less than or equal to $\pi/4$, then

$$\frac{a-d}{1+ad} = \frac{\tan\alpha - \tan\delta}{1+\tan\alpha\tan\delta} = \frac{\tan(\alpha+\pi) - \tan\delta}{1+\tan(\alpha+\pi)\tan\delta} = \tan(\alpha+\pi-\delta) \in [0,1].$$