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The Wheel that leaves Pythagorean Triads in it's Wake

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Introduction

A Pythagorean triad (x, y, u) consists of positive integers x, y, u such that $x^2 + y^2 = u^2$. Geometrically, the integers represent the lengths of the sides of a right-angled-triangle with the hypotenuse u. It immediately follows from Pythagoras' Theorem that the remaining sides cannot have equal length ($\sqrt{2}$ is irrational). Thus, without loss of generality, we suppose that $x < y$ in the following. The $(3, 4, 5)$ Pythagorean triad is well known and it is obvious that if (x, y, u) is a Pythagorean triad then so is (kx, ky, ku) for any integer $k > 0$. (The case $k = 1$ is a primitive Pythagorean triad.) So, $(6, 8, 10)$ is a Pythagorean triad and infinitely many more can be constructed. Geometrically these are similar triangles, but $(5, 12, 13)$ is also a Pythagorean triad and the triangle with sides $(5, 12, 13)$ is not similar to the triangle with sides $(3, 4, 5)$.

In general it can be shown that all primitive Pythagorean triads can be expressed in one of the forms

$$p_1(i,j) := \{(j^2 - i^2, 2ij, j^2 + i^2) : 1 < \frac{j}{i} < 1 + \sqrt{2}\},$$
$$p_2(i,j) := \{(2ij, j^2 - i^2, j^2 + i^2) : \frac{j}{i} > 1 + \sqrt{2}\},$$

where i, j are coprime integers with $j > i$. For example $p_1(1, 2) = (3, 4, 5)$, $p_1(16, 17) = (33, 544, 545)$, $p_2(3, 8) = (48, 55, 73)$, $p_2(1, 24) = (48, 575, 577)$.

This article describes a geometric construction that can be used to generate Pythagorean triads.

(I) The Pythagorean triad in Cell (1)

A wheel of radius = 1 unit rests against a vertical fence of height $= 1+\lambda$ units where $\lambda > 1 \in \mathbb{Q}$, the field of rational numbers. A straight board leans against the wheel with one end at the top of the fence and the other on the ground, as shown in Figure 1. The triangle enclosed by the fence, the straight board and the ground is referred to as Cell (1). The shaded triangle in the figure shown is similar to Cell (1). The length of the sides of Cell (1) are defined as $x_1 = d(O, A_1)$, $y_1 = d(O, B)$, and $u_1 = d(B, A_1)$. The area of this triangle is given by

$$\text{Area} = \frac{1}{2}x_1 y_1$$

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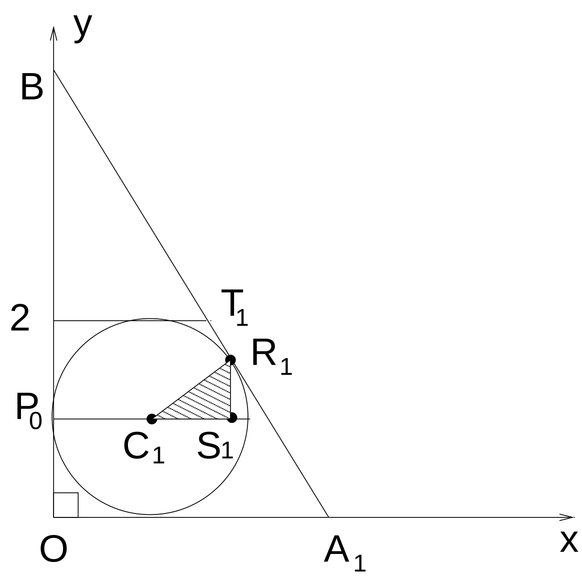

Figure 1: The geometry for Cell (1)

and by

$$\text{Area} = \frac{1}{2}\left(x_1 + y_1 + u_1\right).$$

The first formula is the usual half base times height and the second is half the perimeter times the wheel's radius (in this case unity), which holds in general for a triangle that encloses an inscribed circle. After equating the two expressions for the area, with $y_1 = 1 + \lambda$, and using Pythagoras' Theorem $u_1^2 = x_1^2 + y_1^2$ we find that

$$\begin{aligned} x_1 &= \frac{2\lambda}{\lambda - 1}, \\ u_1 &= \frac{\lambda^2 + 1}{\lambda - 1}. \end{aligned}$$

If we now let $\lambda = \frac{j}{i}$ then it is easy to show that we can write

$$x_1 = \frac{2j}{j - i},$$

$$u_1 = \frac{j^2 + i^2}{i(j - i)},$$

and

$$y_1 = \frac{j^2 - i^2}{i(j - i)}$$

so that

$$i(j - i)(y_1, x_1, u_1) = p_1(i, j) \quad \text{for} \quad \lambda := \frac{j}{i} < 1 + \sqrt{2}$$

and

$$i(j - i)(x_1, y_1, u_1) = p_2(i, j) \quad \text{for} \quad \lambda := \frac{j}{i} > 1 + \sqrt{2}.$$

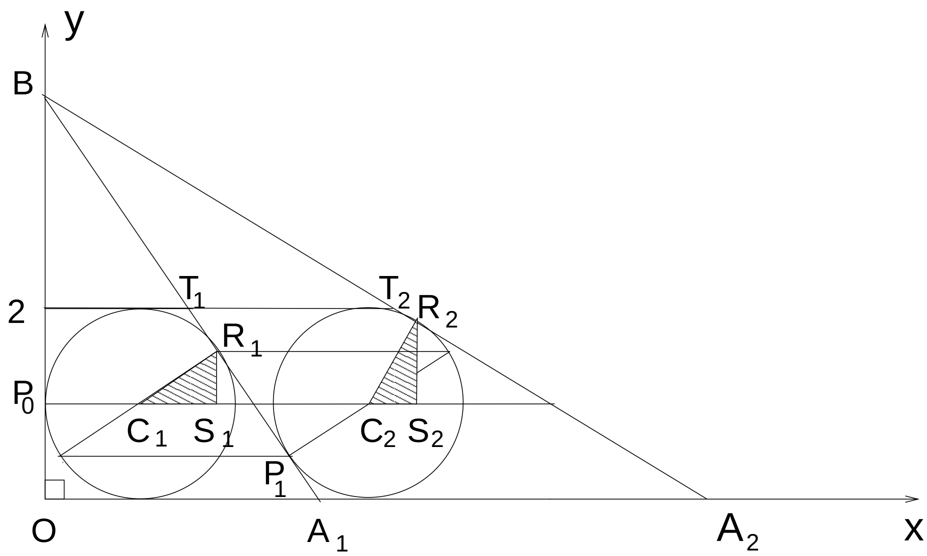

Figure 2: The geometry for Cell (2)

The key to the above construction is that the lengths of the sides of Cell (1) are rational numbers (and scaled up versions are thus integers).

(II) From Cell (1) to Cell (2) to Cell (n)

The straight-board is now temporarily removed and the wheel rolls across until its rear end reaches the position where the straight-board was. This straight-board is then replaced and a second straight-board BA_2 is rested against the wheel's front end as shown in Figure 2. We define Cell (2) as the triangle enclosed by the two straight boards and the ground. The triangle enclosed by the fence, the front straight board and the ground is similar to the shaded triangle in the figure and this yields further Pythagorean triads. Now suppose that this construction of removing and replacing straight boards is continued as the wheel is allowed to roll along the ground, and let Cell (n) denote the triangle enclosed by the two straight boards and the ground after repeating the process n times. At each stage further Pythagorean triads are generated.

General results for Cell (n) are given below.

In Cell(n), the contact points of the wheel with straight boards BA_{n-1} and BA_n are P_{n-1} and R_n respectively. The wheel's centre is C_n, and the intersection of the line $y = 2$ with BA_n is T_n. The following holds true: a)

$$d(C_n, C_{n+1}) = d(T_n, A_n) = \frac{2d(B, A_n)}{\lambda + 1}.$$

b) In reference to$O = (0, 0), C_n, P_n$and$R_n$are related by,

$$\begin{aligned} x_{Pn} + x_{Rn} &= x_{Cn} + x_{Cn+1} \\ y_{Pn} + y_{Rn} &= 2, \end{aligned}$$

where x_{Pn}, y_{Pn} are the x, y coordinates of P_n, and so on.

c) The quadrilateral $C_nP_{n-1}BR_n$ is cyclic, and $C_nP_{n-1}||C_{n-1}R_{n-1}$ hence

$$\angle S_nC_nR_n = \angle S_{n-1}C_{n-1}R_{n-1} + \angle A_{n-1}BA_n = \angle OBA_n$$

and $\triangle S_nC_nR_n|||\triangle OBA_n$.

The perimeter of Cell(n) equals twice its area, hence $x_n := d(O, A_n)$ and $u_n := d(B, A_n)$ are such that

$$\begin{aligned}
x_n - x_{n-1} + u_{n-1} + u_n &= (\lambda+1)(x_n - x_{n-1}) \\
\Rightarrow u_n + u_{n-1} &= \lambda(x_n - x_{n-1}) \\
\text{and} \quad u_n - u_{n-1} &= \frac{u_n^2 - u_{n-1}^2}{u_n + u_{n-1}} \\
&= \frac{x_n^2 + y_1^2 - (x_{n-1}^2 + y_1^2)}{\lambda(x_n - x_{n-1})}) \\
&= \frac{1}{\lambda}\frac{x_n^2 - x_{n-1}^2}{(x_n - x_{n-1})} \\
&= \frac{1}{\lambda}(x_n + x_{n-1})
\end{aligned}$$

which leads to:

$$\begin{aligned}
x_{n+1} &= 2(\frac{\lambda^2+1}{\lambda^2-1})x_n - x_{n-1} \\
u_{n+1} &= 2(\frac{\lambda^2+1}{\lambda^2-1})u_n - u_{n-1} \\
x_n u_{n-1} - x_{n-1}u_n &= \frac{2\lambda(\lambda+1)}{\lambda-1}
\end{aligned}$$

and

$$\begin{aligned}
x_n &= \frac{(\lambda+1)}{2}\left((\frac{\lambda+1}{\lambda-1})^n - (\frac{\lambda-1}{\lambda+1})^n\right) = (\lambda+1)\sinh\left(n\ln(\frac{\lambda+1}{\lambda-1})\right), \\
u_n &= \frac{(\lambda+1)}{2}\left((\frac{\lambda+1}{\lambda-1})^n + (\frac{\lambda-1}{\lambda+1})^n\right) = (\lambda+1)\cosh\left(n\ln(\frac{\lambda+1}{\lambda-1})\right).
\end{aligned}$$

We also have

$$\begin{aligned}
C_n &= \left(\frac{1}{2}\left((\lambda+1)(\frac{\lambda+1}{\lambda-1})^{n-1} - (\lambda-1)(\frac{\lambda-1}{\lambda+1})^{n-1}\right), 1\right) \\
P_n &= \left(\frac{1}{2}\left(\frac{\left(\frac{\lambda+1}{\lambda-1}\right)^{2n} - 1}{\left(\frac{\lambda+1}{\lambda-1}\right)^{2n} + 1}\right)\left((\lambda+1)(\frac{\lambda+1}{\lambda-1})^n + (\lambda-1)(\frac{\lambda-1}{\lambda+1})^n\right), \frac{2}{(\frac{\lambda+1}{\lambda-1})^{2n}+1}\right) \\
R_n &= \left(\frac{1}{2}\left(\frac{\left(\frac{\lambda+1}{\lambda-1}\right)^{2n} - 1}{\left(\frac{\lambda+1}{\lambda-1}\right)^{2n} + 1}\right)\left((\lambda+1)(\frac{\lambda+1}{\lambda-1})^{n-1} + (\lambda-1)(\frac{\lambda-1}{\lambda+1})^{n-1}\right), \frac{2}{(\frac{\lambda-1}{\lambda+1})^{2n}} + 1\right)
\end{aligned}$$

Lastly, the Pythagorean triads in Cell(n) are

$$\begin{aligned}
\frac{i(j^2-i^2)^n}{j+i}(\lambda+1, x_n, u_n) &= \left((j^2-i^2)^n, \frac{(j+i)^{2n}-(j-i)^{2n}}{2}, \frac{(j+i)^{2n}+(j-i)^{2n}}{2}\right) \\
&= p_1(i^{(n)}, j^{(n)}) \quad \text{for} \quad \lambda < \frac{(1+\sqrt{2})^{\frac{1}{n}}+1}{(1+\sqrt{2})^{\frac{1}{n}}-1} \\
\frac{i(j^2-i^2)^n}{j+i}(x_n, \lambda+1, u_n) &= \left(\frac{(j+i)^{2n}-(j-i)^{2n}}{2}, (j^2-i^2)^n, \frac{(j+i)^{2n}+(j-i)^{2n}}{2}\right) \\
&= p_2(i^{(n)}, j^{(n)}) \quad \text{for} \quad \lambda > \frac{(1+\sqrt{2})^{\frac{1}{n}}+1}{(1+\sqrt{2})^{\frac{1}{n}}-1}
\end{aligned}$$

where

$$\begin{aligned}
i^{(n)} &:= \frac{1}{2}\left((j+i)^n - (j-i)^n\right) \\
j^{(n)} &:= \frac{1}{2}\left((j+i)^n + (j-i)^n\right).
\end{aligned}$$

There is a reciprocity between the fence and the ground, in the sense that the transformation $\lambda \to \dfrac{\lambda+1}{\lambda-1}$ implies

$$(x_n, y_1, u_n) \to \left(\frac{\lambda^{2n}-1}{\lambda^{n-1}(\lambda-1)}, x_1, \frac{\lambda^{2n}+1}{\lambda^{n-1}(\lambda-1)}\right) \quad n \geq 1$$

and hence $(x_1, y_1, u_1) \to (y_1, x_1, u_1)$. Accordingly, in the transformed Cell (n), the Pythagorean triads

$$\begin{aligned}
&p_1(i^n, j^n) \qquad \text{for} \quad \lambda < (1+\sqrt{2})^{\frac{1}{n}} \\
&p_2(i^n, j^n) \qquad \text{for} \quad \lambda > (1+\sqrt{2})^{\frac{1}{n}}
\end{aligned}$$

are defined by the scale factor $i^n j^{n-1}(j-i)$. And so the wheel that leaves Pythagorean triads in its wake goes on its merry way.