# 2009 University of New South Wales School Mathematics Competition

## Junior Division – Problems and Solutions

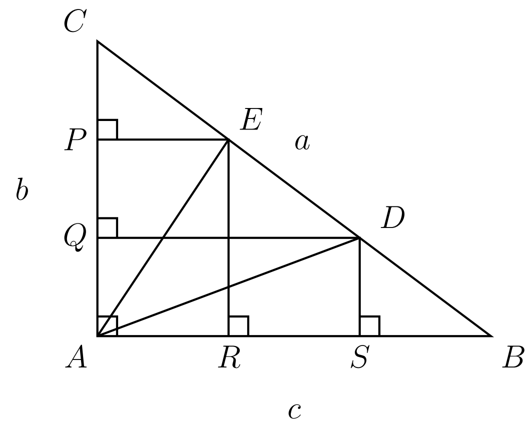
**Problem 1**

Let $ABC$ be a right triangle with right angled at $A$. Let $D, E$ be points on $BC$ with $BD = DE = EC$. Prove that $AD^2 + AE^2 = \frac{5}{9}BC^2$.

**Solution 1(a)**

Let points $R$, $S$ be on $AB$ so that $ER$ and $DS$ are perpendicular to $AB$.

Let $AB = c$, $AC = b$ and $BC = a$.




So $PE \parallel QD \parallel AB$ and $DS \parallel ER \parallel AC$.

Hence since $EC = ED = DB$, then by the Equal Intercepts between Parallel Lines Theorem, we have

$$CP = PQ = QA = \frac{b}{3} \quad \text{and} \quad AR = RS = SB = \frac{c}{3}\ .$$

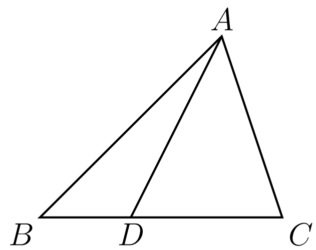
Alternatively, $\triangle PCE \ ||| \ \triangle QCD \ ||| \ \triangle ACB$ and $\triangle SDB \ ||| \ \triangle REB \ ||| \ \triangle ACB$, using equal right angles and corresponding angles in parallel lines, which gives the same result.

Now by Pythagoras' Theorem,

$$\begin{aligned}
AD^2 &= AS^2 + DS^2 = AS^2 + AQ^2 \quad \text{as } DS = AQ \\
&= \left(\frac{2c}{3}\right)^2 + \left(\frac{b}{3}\right)^2 = \frac{4c^2}{9} + \frac{b^2}{9} \\
AE^2 &= AR^2 + RE^2 = AR^2 + AP^2 \quad \text{as } RE = AP \\
&= \left(\frac{c}{3}\right)^2 + \left(\frac{2b}{3}\right)^2 = \frac{c^2}{9} + \frac{4b^2}{9} \\
\therefore \quad AD^2 + AE^2 &= \frac{4c^2}{9} + \frac{b^2}{9} + \frac{c^2}{9} + \frac{4b^2}{9} = \frac{5c^2}{9} + \frac{5b^2}{9} = \frac{5}{9}(c^2 + b^2) \\
&= \frac{5a^2}{9} = \frac{5}{9}BC^2 \,.
\end{aligned}$$

**Solution 1(b)**

Consider the triangle shown.




Apollonius' Theorem states that for any triangle $ABC$ with a point $D$ on $BC$ such that

$$\frac{BD}{DC} = \frac{n}{m} \quad (n \text{ and } m \text{ need only be positive reals!})$$

then $\quad m\,AB^2 + n\,AC^2 = m\,BD^2 + n\,DC^2 + (m+n)\,AD^2.$

In particular if $m = n = 1$, so $D$ is the midpoint of $BC$.

$$\text{Then} \quad AB^2 + AC^2 = BD^2 + DC^2 + 2AD^2 = 2BD^2 + 2AD^2$$

This theorem is easily proved by just dropping an altitude from $A$ to $BC$.

Hence in the original diagram with again $CE = ED = DB = x$, using this midpoint case twice,

$$\begin{aligned}
AD^2 + AC^2 &= 2AE^2 + 2CE^2 = 2AE^2 + 2x^2 \\
\text{and} \quad AB^2 + AE^2 &= 2AD^2 + 2DE^2 = 2AE^2 + 2x^2.
\end{aligned}$$

So adding these equations

$$(AD^2 + AE^2) + (AB^2 + AC^2) = 2(AD^2 + AE^2) + 4x^2$$

but $AB^2 + AC^2 = BC^2 = 9x^2$, so

$$\therefore \quad (AD^2 + AE^2) + 9x^2 = 2(AD^2 + AE^2) + 4x^2$$
$$\text{so} \quad AD^2 + AE^2 = 5x^2 = \frac{5}{9}BC^2.$$

**Problem 2**

A box contains 100 red marbles, 80 green marbles, 60 blue marbles and 40 yellow marbles.

1. What is the smallest number of marbles, which if selected at random, will be guaranteed to contain at least ten pairs of marbles?
2. What is the smallest number of marbles, which if selected at random, will be guaranteed to contain at least ten triplets of marbles?

*In the above a pair of marbles is defined as two marbles of the same colour and a triplet is defined as three of the same colour. No marble may be counted in more than one pair (or triplet).*

**Solution 2**

If $n$ marbles are chosen then $n = r + g + b + y$ with $r$ red marbles, $g$ green marbles, $b$ blue marbles and $y$ yellow marbles.

a) Write $r = 2r_1 + r_2$ etc. with $r_1, r_2$ integers and $r_2 = 0$ or $1$ (Quotient Remainder or Division Algorithm Theorem).

Hence the number of pairs is

$$r_1 + g_1 + b_1 + y_1 = \frac{n}{2} - \left(\frac{r_2 + g_2 + b_2 + y_2}{2}\right) .$$

For $n$ even, this has minimum value

$$\frac{n}{2} - \left(\frac{1+1+1+1}{2}\right) = \frac{n}{2} - 2$$

which is $\geq 10$ iff $n \geq 24$.

For $n$ odd, this has minimum value

$$\frac{n}{2} - \left(\frac{1+1+1+0}{2}\right) = \frac{n}{2} - \frac{3}{2}$$

which is $\geq 10$ iff $n \geq 23$.

Hence the minimum number of marbles required to guarantee $10$ pairs is $23$.

b) This time write $r = 3r_1 + r_2$ etc. where $r_1, r_2$ are integers and $r_2 = 0,\ 1$ or $2$, then the number of triplets is

$$r_1 + g_1 + b_1 + y_1 = \frac{n}{3} - \left(\frac{r_2 + g + 2 + b_2 + y_2}{3}\right)$$

For $n \equiv 0 \pmod 3$, this has minimum value

$$\frac{n}{3} - \left(\frac{2+2+2+0}{3}\right) = \frac{n}{3} - 2$$

which is $\geq 10$ iff $n \geq 36$.

For $n \equiv 1 \pmod 3$, the number of triplets has minimum value

$$\frac{n}{3} - \left(\frac{2+2+2+1}{3}\right) = \frac{n}{3} - \frac{7}{3}$$

which is $\geq 10$ iff $n \geq 37$.

For $n \equiv 2 \pmod 3$, the number of triplets has minimum value

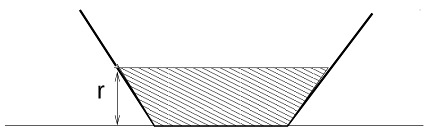
$$\frac{n}{3} - \left(\frac{2+2+2+2}{3}\right) = \frac{n}{3} - \frac{8}{3}$$

which is $\geq 10$ iff $n \geq 38$.

Hence the minimum no. of marbles required to guarantee $10$ triplets is $36$.

**Problem 3**

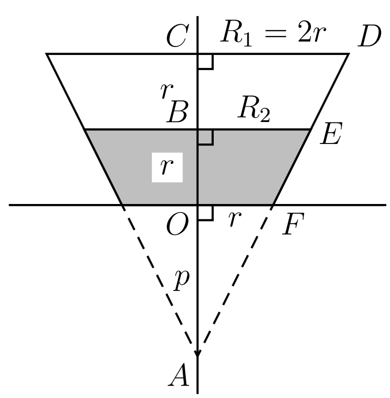
A bowl in the shape of a conical frustum is placed out in the rain at the start of a downpour. At the end of the downpour the water level $r$ is equal to (i) half the height of the bowl, (ii) the radius of the base of the bowl, and (iii) half the radius of the top of the bowl.




The volume of a cone is one-third the area of the circular base times the height. What was the reported rainfall over the catchment area?

**Solution 3**

The figure shows the cross-section of the conical frustum perpendicular to its base containing the axis of the cone,



Let $R_1$ be the radius of the top of the bowl, $R_2$ be the radius of the top of the water level and $r$ be the radius of the bottom of the bowl, and let $p = OA$.

Let $V$ be the volume of water in the bowl.

We are given that $OB = BC = r$ and $R_1 = 2r$.

Since $\triangle\, AOF \; ||| \; \triangle\, ABE \; ||| \; \triangle\, ACD$,

$$\frac{p}{r} = \frac{p+2r}{R_1} = \frac{p+2r}{2r} \quad \text{so} \quad p = \frac{p}{2} + r$$

$$\therefore \quad p = 2r$$

$$\therefore \quad \frac{R_2}{r+p} = \frac{r}{p} \quad \text{so} \quad \frac{R_2}{r+2r} = \frac{1}{2}$$

$$\therefore \quad R_2 = \frac{3r}{2}.$$

$$\begin{aligned} \text{Hence} \quad V &= \frac{1}{3}\,\pi\, R_2^2(p+r) - \frac{1}{3}\,\pi\, r^2\, p \\ &= \frac{\pi}{3}\left(\frac{3r}{2}\right)^2 \cdot 3r - \frac{\pi}{3} \cdot r^2 \cdot 2r \\ &= \frac{\pi r^3}{3}\left(\frac{27}{4} - 2\right) = \frac{\pi r^3}{3} \cdot \frac{19}{4} \\ &= \frac{19\pi r^3}{12}. \end{aligned}$$

Rainfall is measured as the volume of water per unit area on which the water fell (assumed perpendicularly) over some time period. The water in the bowl fell on the circular cross-section at the top of the bowl, which has area $\pi R_1^2 = \pi(2r)^2 = 4\pi r^2$, hence the rainfall is

$$\frac{19\pi r^3}{12} \cdot \frac{1}{4\pi r^2} = \frac{19r}{48}\,.$$

**Problem 4**

There are many ways to substitute digits for letters in the expression

$$CAT \quad + \quad EMU \quad = \quad LION,$$

with different letters representing different digits, to obtain a valid equation. Prove that in every solution, $LION$ is a multiple of $9$. Is there any higher number than $9$ for which this is still true?

**Solution 4**

We wish to show that

$$LION \equiv 0 \pmod 9.$$

Without loss of generality we write

$$\begin{aligned} CAT &\equiv a \mod 9 \\ EMU &\equiv b \mod 9 \\ LION &\equiv c \pmod 9), \end{aligned}$$

and using the digit sum rule

$$\begin{aligned} C+A+T &\equiv a \pmod 9 \\ E+M+U &\equiv b \pmod 9 \\ L+I+O+N &\equiv c \pmod 9). \end{aligned}$$

(The digit sum rule can be seen by expanding

$$CAT = 100C + 10A + T = 99C + C + 9A + A + T$$

so that $CAT \pmod 9) \equiv C + A + T \pmod 9)$ etc.)

Using the given equation

$$CAT + EMU = LION$$

we have

$$a + b \equiv c \pmod 9).$$

Now note that there are ten different letters in total so that

$$\begin{aligned} C+A+T+E+M+U+L+I+O+N &= 0+1+2+3+4+5+6+7+8+9 \\ &= 45 = 9 \cdot 5, \end{aligned}$$

and hence

$$a + b + c \equiv 0 \pmod 9).$$

Combining the results, $a+b \equiv c \pmod 9)$ and $a+b+c \equiv 0 \pmod 9)$, we deduce that $2c \equiv 0 \pmod 9)$, and hence $c \equiv 0 \pmod 9)$ as required.

For the second part note that $1089 = 324+765$ and $1098 = 346+752$ are both $LION$s and their greatest common divisor is 9.

**Problem 5**

Let $n$ be a positive integer and let $S_n = 1^n + 2^n + 3^n + 4^n$.

1. Find $S_1, S_2, S_3, S_4$.

2. Show that $n^5$ has the same last digit (units digit) as $n$ does.

3. Prove that 10 is a factor of $S_n$ unless $n$ is a multiple of 4.

**Solution 5**

1. $S_1 = 10, S_2 = 30, S_3 = 100, S_4 = 354$

2. One can check $n^5 \equiv n \pmod{10}$ for $n = 0, 1, 2, 3, \cdots, 9$ In fact we need only check this for $n = 0, 1, 2, 3, 4, 5$ as $6 \equiv -2$ etc. and we have odd powers here.

   Hence this is true for all integers as any integer $n$ is $\equiv 0, 1, 2, 3 \cdots, 9 \pmod{10}$ and if $n \equiv m \pmod{10}$ then $n^p \equiv m^p \pmod{10}$ for any positive integer $p$, OR write $n = 10k + r$ with $k$ a non-negative integer and $r = 0, 1, 2, 3, \cdots 8$ or $9$ (Division Algorithm or Quotient-Remainder Theorem), then by the Binomial Theorem,

$$\begin{aligned} n^5 &= (10k + r)^5 \\ &= (10k)^5 + 5 \cdot (10k)^4 r + 10 \cdot (10k)^3 r^2 + 10 \cdot (10k)^2 r^3 + 5 \cdot (10k) r^4 + r^5 \\ &\equiv r^5 \equiv r \equiv n \pmod{10}. \end{aligned}$$

3. For all integers $n$, and all non-negative integers $k$ and positive integers $r$ (and for $r = 0$ if $k > 0$), we have

$$n^{4k+r} \equiv n^r \pmod{10}$$

by induction on $k$. Since it is trivial for $k = 0$ and if true for some $k$, then

$$\begin{aligned} n^{4(k+1)+r} &\equiv n^{4k+4+r} \equiv n^{5+4k+r-1} \equiv n^5 \cdot n^{4k+r-1} \\ &\equiv n \cdot n^{4k+r-1} \equiv n^{4k+r} \equiv n^r \pmod{10}. \end{aligned}$$

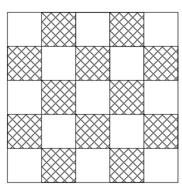
Hence for $r = 1, 2, 3, 4$ and $k$ a positive integer, we have mod $10$,

$$\begin{aligned} S_{4k+r} &\equiv 1^{4k+r} + 2^{4k+r} + 3^{4k+r} + 4^{4k+r} \\ &\equiv 1^r + 2^r + 3^r + 4^r \equiv S_r \equiv \begin{cases} 0 & \text{if } r = 1, 2, 3 \\ 4 & \text{if } r = 4. \end{cases} \end{aligned}$$

Hence $10$ divides $S_n$ if and only if $4$ does not divide $n$.

**Problem 6**

A quadrophage is an insect which eats checkerboards (square boards tiled with squares of alternating light and dark colours). A $5 \times 5$ checkerboard is shown below.



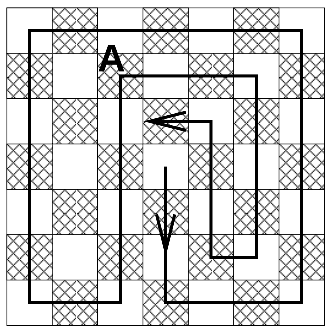
The quadrophage starts off in a given direction and always eats one square before proceeding to the next. As long as there is a square to be eaten it will always proceed in a straight line along a row or column of the board. If it encounters the edge of the board, or a square it has already eaten, it turns through a right angle (either left or right) and keeps going. If it reaches a point with no neighbouring uneaten squares it must stop.

1. If the quadrophage starts on the middle square of a $7 \times 7$ checkerboard, what is the maximum number of squares it can eat? Draw a diagram of the path taken.

2. The quadrophage jumps on to a random square of a large checkerboard and eventually finds that it has eaten the whole board. On which squares might it have started?

**Solution 6**

1. The case in which the quadrophage first moves down and then to the right is shown in the figure on page 38. Note that the quadrophage has no choice until it gets to the square labelled A in this figure.

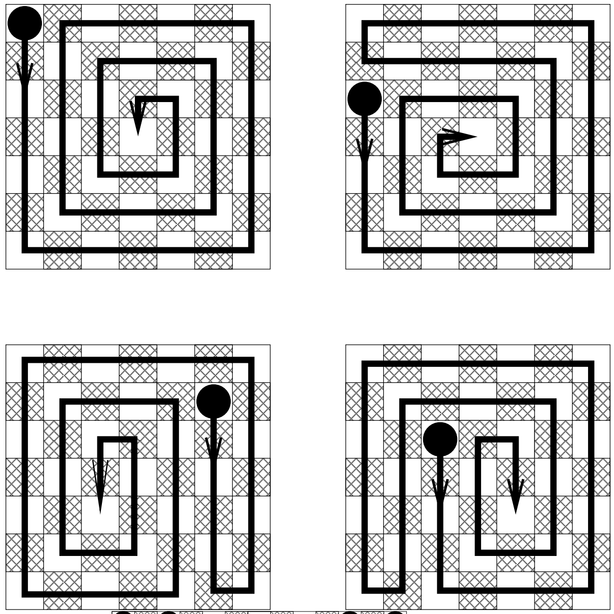


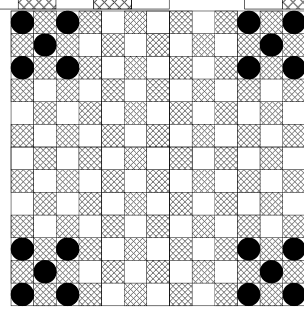

   Thus except for rotations and reflections, there are only 2 possible paths starting at the middle square, one path misses 12 squares so 37 are eaten, the other path misses 5 squares so 44 are eaten. The maximal number of squares that can be eaten is thus 44.

2. It is sufficient to consider a $7 \times 7$ checkerboard for the general result. We begin by identifying starting squares, that result in all squares being eaten, under the restriction that the quadrophage first moves down and then moves to the right at the bottom of the board. The complete set of starting squares that result in all squares being eaten can then be arrived at by considering reflections and rotations of the board.

   All possibilities in which all squares are eaten with the quadrophage first moving downwards and then to the right are shown in the figure on page 39.

   After considering reflections and rotations we then arrive at five possible starting points within each $3 \times 3$ corner region. The complete set of starting points is then as shown by filled circles in the $13 \times 13$ checkerboard below.





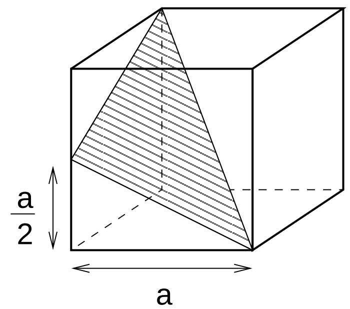
## Senior Division – Problems and Solutions

**Problem 1**
See Problem 6 in the Junior Competition.
**Solution 1**
See Problem 6 solution in the Junior Competition.
**Problem 2**
A triangle has one side that is the space diagonal (or long diagonal) of a cube and the other two sides lie on the surface of the cube whose side has length $a$.

1. What is the smallest perimeter that such a triangle can have?

2. What is the smallest area that such a triangle can have?

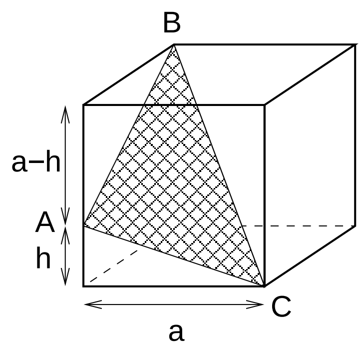
**Solution 2**
The triangle shown in the figure has both the smallest perimeter and the smallest area given the constraints.




1. It is easy to see that the given triangle has the smallest perimiter. Flatten out the faces containing the edges of the triangle, the resultant straight line is the shortest distance between the two points connecting the space diagonal. Some straightforward applications of Pythagoras's Theorem then yield the result

$$\ell = 2\sqrt{a^2 + \left(\frac{a}{2}\right)^2} + \sqrt{3}a = (\sqrt{3} + \sqrt{5})a.$$

2. Consider the geometry shown with the vertices of the triangle labelled $A$, $B$, $C$.




Define the semi-perimeter $s = \frac{AB+BC+CA}{2}$ and the area from Heron's formula

$$\text{Area} \quad = \sqrt{s(s - AB)(s - BC)(s - CA)}.$$

Use Pythagoras's Theorem to write

$$\begin{aligned} AB &= \sqrt{a^2 + (a - h)^2}, \\ BC &= \sqrt{3}a, \\ CA &= \sqrt{a^2 + h^2}, \end{aligned}$$

and then after a little algebra

$$\text{Area} \quad = \frac{a}{\sqrt{2}}\sqrt{a^2 + h^2 - ah}.$$

Note that $\frac{d(\text{Area})}{dh} = 0$ when $h = \frac{a}{2}$ and $\frac{d^2(\text{Area})}{dh^2} > 0$ when $h = \frac{a}{2}$ so that the minimum area is Area $= \frac{\sqrt{6}}{4}a^2$.

An alternate method is to consider the minimum distance from the vertex on the edge of the cube to the space diagonal. This minimum distance is $s = \frac{a}{\sqrt{2}}$ from which the mimimum area Area $= \frac{1}{2}\frac{a}{\sqrt{2}}\sqrt{3}a = \frac{\sqrt{6}}{4}a^2$ follows.

**Problem 3**

Suppose that $f(0) = 1$ and $a > 0$ is a fixed real number. Show that $f(x) = 1$ is the only continuous function $f : \mathbb{R} \to \mathbb{R}$ that satisfies

$$2f(x-y) = f(a-y)f(x) + f(a-x)f(y)$$

for all $x, y \in \mathbb{R}$.

**Solution 3**

Put $x = y$ then

$$f(0) = \frac{1}{2}f(a-x)f(x) + \frac{1}{2}f(a-x)f(x)$$

so that

$$f(x) = \frac{1}{f(a-x)}.$$

Note that $f(a) = 1/f(0) = 1$ follows from this.

Now consider $y = 0$ then

$$f(x) = \frac{1}{2}f(a)f(x) + \frac{1}{2}f(a-x)f(0)$$

and hence

$$f(x) = \frac{1}{2}f(x) + \frac{1}{2}f(a-x)$$

so that

$$f(x) = f(a-x).$$

Combining the above results we now obtain

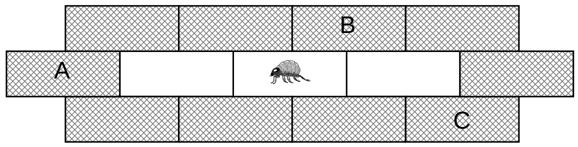
$$f(a-x) = \frac{1}{f(a-x)}$$

from which it follows that $f(a-x) = \pm 1$ for all $x \in \mathbb{R}$ and thus $f(x) = \pm 1$ for all $x \in \mathbb{R}$. We now reject the possibility that $f(x) = -1$ since $f(0) = +1$ so that there would be a jump discontinuity at $x$ corresponding to the smallest value of $|x|$ where $f(x) = -1$. Note that it is obvious by inspection that $f(x) = 1$ is a solution. Here we have proved it is the only continuous solution.

**Problem 4**

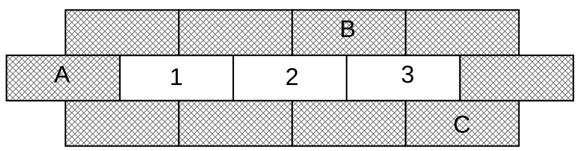
A flea starts on the brick in the centre of a portion of brickwall (as shown) and hops from one brick to any one of the neighbouring six bricks with equal probability. If it hops onto one of the shaded bricks it is promptly squashed. Otherwise it hops again to one of the neighbouring bricks with equal probability. The flea continues hopping in this fashion and is allowed to visit the same white brick more than once.

What is the probability $p_A$ that the flea is squashed on the shaded brick labelled $A$? What is the probability $p_B$ that the flea is squashed on the shaded brick labelled $B$? What is the probability $p_C$ that the flea is squashed on the shaded brick labelled $C$?




**Solution 4**

It is convenient to number the white bricks $1, 2, 3$ from left to right, and let $E_n$ denote the expectation that the flea arrives on the $n$th brick (after any number of hops) before it hops onto one of the shaded bricks.

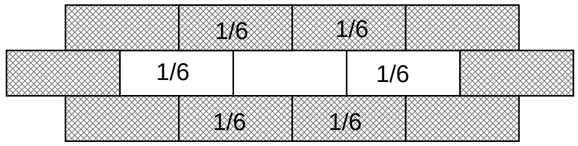


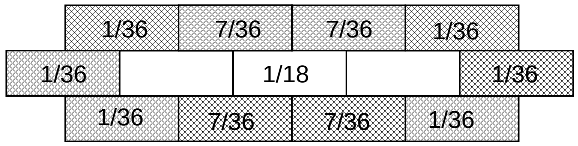

From each white brick the flea can hop to any one of the six neighbouring bricks from which we deduce

$$\begin{aligned} p_A &= \frac{1}{6}E_1 \\ p_B &= \frac{1}{6}E_2 + \frac{1}{6}E_3 \\ p_C &= \frac{1}{6}E_3 \end{aligned}$$

and $E_1 = \frac{1}{6}E_2, \quad E_3 = \frac{1}{6}E_2$. It is clear from the symmetry of the problem that $E_1 = E_3$ so that we can express each of the probabilities in terms of $E_2$, as follows: $p_A = \frac{1}{36}E_2$, $p_B = \frac{7}{36}E_2$, and $p_C = \frac{1}{36}E_2$. Given that the probabilities must sum up to unity we also have $2p_A + 4p_C + 4p_B = 1$ so that $\left(\frac{2}{36} + \frac{4}{36} + \frac{28}{36}\right) E_2 = 1$. We now solve for $E_2 = \frac{36}{34}$ so that $E_1 = \frac{6}{34}$ and then $p_A = \frac{1}{34}$, $p_B = \frac{7}{34}$ and $p_C = \frac{1}{34}$.

An alternate method of solution is to consider the probabilities for the location of the flea after the first two hops as shown:




We can now deduce that after many hops

$$p_C = \frac{1}{36} + \left(\frac{1}{18}\right)\frac{1}{36} + \left(\frac{1}{18}\right)^2\frac{1}{36} + \left(\frac{1}{18}\right)^3\frac{1}{36} + \dots$$

and by summing the geometric series

$$p_C = \frac{1}{36}\left(\frac{1}{1-\frac{1}{18}}\right) = \frac{1}{34}.$$

Similarly we find $p_A = \frac{1}{34}$ and $p_B = \frac{7}{34}$.

**Problem 5**

Consider a sequence of integers $a_1, a_2, a_3, \dots$ with $a_n = a_{n-1} - a_{n-2}$ if $n \geq 3$. Find the sum of the first 2009 terms in the sequence given that the sum of the first 2007 terms is 2008 and the sum of the first 2008 terms is 2007.

**Solution 5**

Starting with $a_1, a_2$ we find that $a_3 = a_2 - a_1$, $a_4 = -a_1$, $a_5 = -a_2$, $a_6 = a_1 - a_2$, $a_7 = a_1$, $a_8 = a_2$ so that

$$a_n = \begin{cases} a_1 & \text{if} \quad n = 1 + 6k \\ a_2 & \text{if} \quad n = 2 + 6k \\ a_2 - a_1 & \text{if} \quad n = 3 + 6k \\ -a_1 & \text{if} \quad n = 4 + 6k \\ -a_2 & \text{if} \quad n = 5 + 6k \\ a_1 - a_2 & \text{if} \quad n = 6 + 6k, \end{cases}$$

where $k = 1, 2, 3, \dots$. From the above it is straightforward to compute the sum of the first $n$ terms which we denote by $s_n$. This yields

$$s_n = \begin{cases} a_1 & \text{if} \quad n = 1 + 6k, \\ a_1 + a_2 & \text{if} \quad n = 2 + 6k, \\ 2a_2 & \text{if} \quad n = 3 + 6k, \\ 2a_2 - a_1 & \text{if} \quad n = 4 + 6k, \\ a_2 - a_1 & \text{if} \quad n = 5 + 6k, \\ 0 & \text{if} \quad n = 6 + 6k. \end{cases}$$

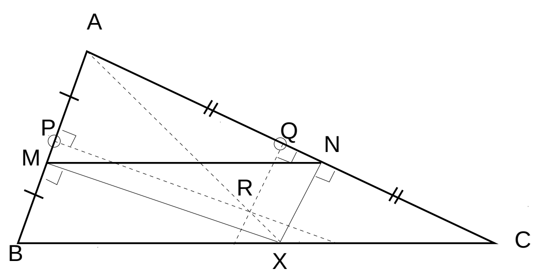
Now note that $2007 = 3 + 6(334), 2008 = 4 + 6(334), 2009 = 5 + 6(334)$ hence $s_{2007} = 2a_2 = 2008$ and $s_{2008} = 2a_2 - a_1 = 2007$, from which we deduce $a_2 = 1004$, $a_1 = 1$ and then $s_{2009} = a_2 - a_1 = 1003$.

**Problem 6**

Let $ABC$ be a triangle with acute angles at $B$ and $C$. For any $X$ on $BC$, let $M$ and $N$ be the feet of the perpendiculars from $X$ to $AB$ and $AC$. Show how to find $X$ so that $MN$ is parallel to $BC$.

**Solution 6**

Let $P, Q$ be the midpoints of $AB, AC$ and let $R$ be the circumcentre of the triangle; let $X$ be the intersection of $AR$ and $BC$.




We note that:

$$\begin{aligned}
\triangle APQ \;|||\; \triangle ABC \qquad & \text{so that} \quad \frac{AP}{AQ} = \frac{AB}{AC}, \\
\triangle AQR \;|||\; \triangle ANX \qquad & \text{so that} \quad \frac{AQ}{AN} = \frac{AR}{AX}, \\
\triangle APR \;|||\; \triangle AMX \qquad & \text{so that} \quad \frac{AP}{AM} = \frac{AR}{AX}.
\end{aligned}$$

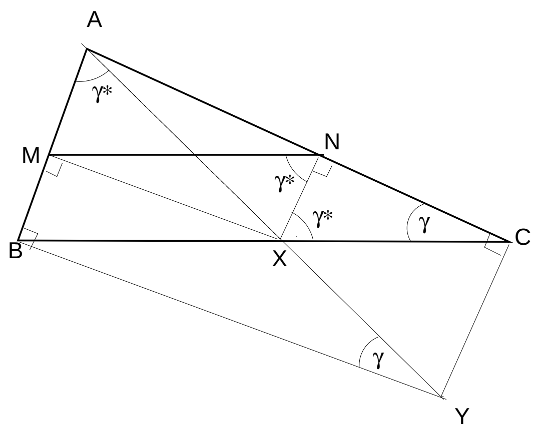
From the above we now deduce that

$$\frac{AM}{AB} = \frac{AN}{AC}$$

so that $\triangle AMN \;|||\; \triangle ABC$ and $MN$ is parallel to $BC$.

An alternate construction that yields the same result is shown below.

Let $Y$ be the unique intersection of the perpendicular to $AB$ through $B$ and the perpendicular to $AC$ through $C$. The $X$ is the intersection of $AY$ and $BC$. The points $ABYC$ are concyclic so that $\angle BYA = \gamma$ and then $\angle BAY = \gamma^*$. The points $AMXN$ are also concyclic so that now $\angle MNX = \gamma^*$. therefore $MN \parallel BC$ (alternate angles are equal).


A
γ*
M
N
γ*
γ*
γ
B
X
C
γ
Y