Solutions to Problems 1291-1300

Q1291 Show that there do not exist three primes x, y and z satisfying

$$x^2 + y^3 = z^4.$$

ANS: (Correct solution by J.C. Barton, Victoria)

Assume that such primes exist. First we note that either one and only one of x, y and z is even, or all three are even, because otherwise one side is even whereas the other side is odd. The only even prime being 2, the case that x, y and z are all even is not possible, because otherwise $2^2 + 2^3 = 2^4$; contradiction. There remain 3 cases to consider.

Case 1: only $z = 2$. Then $x^2 + y^3 = 16$, implying $0 \le y^3 \le 16$. This in turn yields $y = 2$; contradiction!

Case 2: only $y = 2$. Then $(z^2 + x)(z^2 - x) = 8$, resulting in

$$z^2 + x = 4 \quad \text{and} \quad z^2 - x = 2,$$

or

$$z^2 + x = 8 \quad \text{and} \quad z^2 - x = 1,$$

implying $z^2 = 3$, or $z^2 = 9/2$; contradiction!

Case 3: only $x = 2$. Then $y^3 = (z^2 + 2)(z^2 - 2)$. Since y is a prime, y^3 has 4 divisors 1, y, y^2 and y^3, and we have either

$$z^2 - 2 = y \quad \text{and} \quad z^2 + 2 = y^2,$$

or

$$z^2 - 2 = 1 \quad \text{and} \quad z^2 + 2 = y^3.$$

The first possibility results in $y^2 - y = 4$ and the second results in $y^3 - 1 = 4$. Both cases lead to a contradiction.

Therefore, there do not exist primes satisfying the given equation.

Q1292 Prove that for all a, b, c, and d satisfying $0 \le a, b, c, d \le 1$, there holds

$$\frac{a}{b+c+d+1} + \frac{b}{c+d+a+1} + \frac{c}{d+a+b+1} + \frac{d}{a+b+c+1} + (1-a)(1-b)(1-c)(1-d) \le 1.$$

ANS: Without loss of generality we can assume that

$$0 \le a \le b \le c \le d \le 1$$

. First note that by Cauchy's inequality there holds

$$\begin{aligned}
& (a+b+c+1)(1-a)(1-b)(1-c) \\
\leq & \left(\frac{(a+b+c+1)+(1-a)+(1-b)+(1-c)}{4}\right)^4 \\
= & 1,
\end{aligned}$$

so that

$$(1-a)(1-b)(1-c) \leq \frac{1}{a+b+c+1},$$

or

$$(1-a)(1-b)(1-c)(1-d) \leq \frac{1-d}{a+b+c+1}.$$

Now consider the left-hand side of the inequality to be proved. Noting that $a, b, c \leq d$ we have

$$\begin{aligned}
LHS & \leq \frac{a}{a+b+c+1} + \frac{b}{a+b+c+1} + \frac{c}{a+b+c+1} + \frac{d}{a+b+c+1} \\
& + (1-a)(1-b)(1-c)(1-d) \\
& \leq \frac{a+b+c+d}{a+b+c+1} + \frac{1-d}{a+b+c+1} \\
& = 1 = RHS.
\end{aligned}$$

Q1293 Suppose that $u = \cot(\pi/8)$ and $v = \operatorname{cosec}(\pi/8)$. Prove that u satisfies a quadratic and v a quartic equation with integral coefficients and with leading coefficients 1.

ANS: (Correct solution by J.C. Barton, Victoria)

Using the double-angle formula for $\tan 2x$ we have

$$1 = \tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}},$$

so that

$$\frac{2/u}{1-1/u^2} = 1,$$

or

$$\frac{2u}{u^2-1} = 1,$$

implying $u^2 - 2u - 1 = 0$. Meanwhile, the double-angle formula for $\sin 2x$ gives

$$\frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} = 2\sin\frac{\pi}{8}\cos\frac{\pi}{8},$$

so that, by squaring both sides and using $\cos^2 x = 1 - \sin^2 x$,

$$\frac{4}{v^2}\left(1-\frac{1}{v^2}\right) = \frac{1}{2}.$$

Multiplying both sides by $2v^4$ gives

$$8(v^2-1)=v^4,$$

or $v^4-8v^2+8=0$.

Q1294 Let a and b be two sides of a triangle, and α and β be two angles opposite these sides, respectively. Prove that

$$\frac{a+b}{a-b}=\frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}.$$

ANS: (Correct solution by J.C. Barton, Victoria)

By using the law of sines

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=2R$$

where R is the radius of the circumscribed circle, we have

$$\frac{a+b}{a-b}=\frac{2R(\sin\alpha+\sin\beta)}{2R(\sin\alpha-\sin\beta)}=\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}.$$

By using the addition formulas for $\sin(x+y)$ and $\cos(x+y)$ we can prove that

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\sin\alpha-\sin\beta=2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}.$$

Therefore,

$$\frac{a+b}{a-b}=\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2}}\,\frac{\cos\frac{\alpha-\beta}{2}}{\sin\frac{\alpha-\beta}{2}}=\frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}.$$

Q1295 Assume that the following information about a triangle is known: the radius R of the circumscribed circle, the length c of one side, and the ratio a/b of the lengths of the other two sides. Determine all three sides and angles of this triangle.

ANS: First note that $c\le 2R$. Let α, β and γ be the angles opposite sides a, b and c, respectively. Then by using the law of sines we have

$$\sin\gamma=\frac{c}{2R}.$$

If $c<2R$, there are two possible values for γ. If $c=2R$ we have $\gamma=\pi/2$.

Having found γ we obtain $(\alpha+\beta)/2$ by

$$\frac{\alpha+\beta}{2}=\frac{180^\circ-\gamma}{2}.$$

It follows from **Q1294** that

$$\tan\frac{\alpha-\beta}{2} = \frac{a-b}{a+b}\,\tan\frac{\alpha+\beta}{2} = \frac{(a/b)-1}{(a/b)+1}\,\tan\frac{\alpha+\beta}{2},$$

so that $\frac{\alpha-\beta}{2}$ is determined. Hence α and β can be obtained from

$$\alpha = \frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2} \quad \text{and} \quad \beta = \frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}.$$

Finally, the law of sines gives

$$a = \frac{c\sin\alpha}{\sin\gamma} \quad \text{and} \quad b = \frac{c\sin\beta}{\sin\gamma}.$$

Q1296 (Suggested by Dr. Panagiote Ligouras, Leonardo da Vinci High School, Noci, Bari, Italy)

Let a, b and c be the sides, and m_a, m_b and m_c be the medians of a triangle ABC. Prove that

$$m_a^2 = \frac{1}{4}(2b^2+2c^2-a^2), \quad m_b^2 = \frac{1}{4}(2c^2+2a^2-b^2), \quad m_c^2 = \frac{1}{4}(2a^2+2b^2-c^2).$$

ANS: (Correct solution by J.C. Barton, Victoria)

Let M be the midpoint of BC. Then by using the cosine rule for two triangles AMC and AMB we obtain

$$b^2 = m_a^2 + \frac{1}{4}a^2 - am_a\cos\angle AMC$$
$$c^2 = m_a^2 + \frac{1}{4}a^2 - am_a\cos\angle AMB.$$

Since $\cos\angle AMB = -\cos\angle AMC$, by adding the above equations we obtain the desired formula for m_a. Similar arguments hold for m_b and m_c.

Q1297 (Suggested by Dr. Panagiote Ligouras, Leonardo da Vinci High School, Noci, Bari, Italy)

Let a, b and c be the sides, and m_a, m_b and m_c be the medians of a triangle ABC. Prove or disprove that

$$27(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2) \le 64(m_a^4+m_b^4+m_c^4)(m_a^2+m_b^2+m_c^2).$$

ANS: First we note from **Q1296** that

$$m_a^2+m_b^2+m_c^2 = \frac{3}{4}(a^2+b^2+c^2) \quad \text{and} \quad m_a^4+m_b^4+m_c^4 = \frac{9}{16}(a^4+b^4+c^4). \qquad (0.1)$$

Now by using the Cauchy-Schwarz inequality we have

$$a^2b + b^2c + c^2a \le \sqrt{a^4 + b^4 + c^4}\,\sqrt{b^2 + c^2 + a^2}, \tag{0.2}$$

and

$$ab^2 + bc^2 + ca^2 \le \sqrt{a^2 + b^2 + c^2}\,\sqrt{b^4 + c^4 + a^4}. \tag{0.3}$$

Combining (0.1)–(0.3) we obtain

$$\begin{aligned}(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) &\le (a^4 + b^4 + c^4)(a^2 + b^2 + c^2)\\ &= \frac{16}{9}(m_a^4 + m_b^4 + m_c^4)\,\frac{4}{3}(m_a^2 + m_b^2 + m_c^2),\end{aligned}$$

yielding the desired inequality.

Q1298 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(a+b) + f(b+c) + f(c+a) \ge 3f(a+2b+3c) \quad \text{for all } a, b, c \in \mathbb{R}.$$

ANS: Consider an arbitrary $x \in \mathbb{R}$ and put $a = x$ and $b = c = 0$ in the given inequality. Then $2f(x) + f(0) \ge 3f(x)$, implying $f(x) \le f(0)$. Put $a = b = x/2$ and $c = -x/2$. Then $f(x) + 2f(0) \ge 3f(0)$, implying $f(x) \ge f(0)$. Hence $f(x) = f(0)$ for all $x \in \mathbb{R}$, i.e. f is a constant function. Obviously, any constant function satisfies the given inequality.

Editor's note: Here we have shown that if f satisfies the given inequality then f is constant. This means we have found ALL functions f.

Q1299 Let f be a function satisfying each of the following

1. For all real numbers x and y, there holds

$$f(x+y) + f(x-y) = 2f(x)f(y). \tag{0.4}$$

2. There exists a real number a such that $f(a) = -1$.

Prove that f is periodic.

ANS: (Correct solution by J.C. Barton, Victoria)

By putting $x = a$ and $y = 0$ in (0.4) we deduce $f(0) = 1$. By letting $x = y = a/2$ in (0.4) we deduce

$$f(a) + 1 = 2\left(f(\frac{a}{2})\right)^2,$$

implying $f(a/2) = 0$. Hence, for any $x \in \mathbb{R}$,

$$f(x + \frac{a}{2}) + f(x - \frac{a}{2}) = 2f(x)f(a/2) = 0,$$

so that

$$f(x + \frac{a}{2}) = -f(x - \frac{a}{2}) \quad \text{for all } x \in \mathbb{R}.$$

Using this identity twice yields

$$f(x + 2a) = -f(x + a) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

Therefore, f is periodic with period $2a$.

Q1300 Find all polynomials $p(x)$ satisfying

$$(x-16)p(2x) = 16(x-1)p(x) \quad \text{for all } x \in \mathbb{R}. \tag{0.5}$$

ANS: (Correct solution by J.C. Barton, Victoria)

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_n \neq 0$. By equating the leading terms on both sides of (0.5) we obtain $2^n a_n = 16a_n$. Since $a_n \neq 0$ it follows that $n = 4$. Hence $p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. There are two ways to find p.

Method 1: By substituting $p(x) = a_4x^4+a_3x^3+a_2x^2+a_1x+a_0$ into (0.5) and equating the coefficients we obtain the following relations in the coefficients

$$\begin{aligned} 8a_3 - 256a_4 &= 16a_3 - 16a_4 \\ 4a_2 - 128a_3 &= 16a_2 - 16a_3 \\ 2a_1 - 64a_2 &= 16a_1 - 16a_2 \\ a_0 - 32a_1 &= 16a_0 - 16a_1. \end{aligned}$$

Let $a_0 = t$ be an arbitrary real number. Then

$$a_1 = -\frac{15}{16}t, \quad a_2 = \frac{35}{128}t, \quad a_3 = -\frac{15}{512}t, \quad a_4 = \frac{15}{14336}t.$$

These coefficients define all possible polynomials

$$p(x) = t\left(\frac{15}{14336}x^4 - \frac{15}{512}x^3 + \frac{35}{128}x^2 - \frac{15}{16}x + 1\right),$$

where t is any real number.

Method 2: By substituting successively $x = 1$ and $x = 16$ into (0.5) we have $p(2) = 0$ and $p(16) = 0$. Next substituting $x = 2$ gives $p(4) = 0$. Finally, substituting $x = 4$ gives $p(8) = 0$. Therefore

$$p(x) = a(x-2)(x-4)(x-8)(x-16),$$

where a is any real number.