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Marx and Mathematics

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I was rather idly browsing the shelves of one of the University of Melbourne's libraries, when a volume caught my eye. It was entitled *Les manuscrits mathématiques de Marx* (*Marx's mathematical manuscripts*). My curiosity was aroused and I readily learned that the Marx in question was indeed Karl Marx, *the* Karl Marx, advocate of the overthrow of capitalism. It was a complete surprise to me that Marx had had any interest at all in mathematics, let alone written on the subject. However he did indeed write a number of papers on aspects of mathematics, although none of these writings were published in his lifetime; in fact he may never even have had publication in mind when he prepared them.

However, his various writings on mathematics are now available. The background to this development is intricate but relevant. Marx wrote in German, and a tentative date of 1881 has been ascribed to his mathematical writings, although his correspondence actually makes it clear that he had been working on them for some considerable time before this date. The manuscripts passed on Marx's death (1883) into the keeping of his associate Friedrich Engels (co-author with Marx of *The Communist Manifesto*), who may have had plans to publish them. He did not get around to it. Engels himself died in 1895, and the entire collection of papers by Marx and Engels passed into the hands of the German Social Democrats. There they stayed for a long time. However, the Russian revolution sparked an interest in Marx's writing and Lenin was anxious that every bit of it should see the light of day.

But, Lenin in his turn died and the mathematical manuscripts had still not been published. Eventually a Russian translation was produced (in 1933). After that, there was rather more interest. A copy of the original German came into the hands of Dirk Struik (1894–2000), an eminent historian of mathematics. Struik had considerable sympathy with Marxist thought (which later led to his landing in a lot of trouble during the McCarthy era), and it may have been this sympathy that led to his being entrusted with this material. He wrote a paper on the subject and this first appeared in 1948. This paper remains the most authoritative account of the matter, although he is to my mind over-generous in the significance he ascribes to Marx's contributions.

Now there are many available versions of the manuscripts. I have already mentioned the French one which (probably) appeared in 1985. A somewhat earlier date (1983) saw the printing of an English translation, and there have been others. Apparently a Japanese translation is extant, as are two separate Chinese ones, a Portuguese and a Greek. The original German is also available: it was published in the then Soviet

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Union in 1968. Quite possibly there are yet others. Although the manuscripts were neglected for almost 50 years, this neglect has now been well and truly redressed!

But what do the manuscripts say? And are they important?

Well, here opinions differ. The principal concern they address is the logical foundation of calculus. This had been a vexed issue ever since the pioneering work of Newton and Leibniz. I devoted my column in *Parabola* **41(1)** (2005) and an earlier one in *Function* **14(3)** (1990) to aspects of it. Early versions of calculus employed 'differentials' (or 'fluxions' as Newton called them). These were quantities of infinitesimal extent – zero, but not really zero. This basic contradiction beset calculus for over a century and evoked a scornful response from George Berkeley, a bishop and a slightly younger contemporary of Newton:

"And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them ghosts of departed quantities?"

As I recounted in my earlier column, the concept of a differential was ultimately rendered respectable by Abraham Robinson in 1961. However, long before this, calculus had been placed on a logically secure footing by the work of Augustin-Louis Cauchy (1789-1857). In his 1821 *Cours d'analyse,* Cauchy developed a new approach, one that dispensed with the idea of the differential altogether. The new look calculus employed instead the notion of *a limit*. I will illustrate this first by a non-calculus example. Suppose we want the value of the sum

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + ...$$

Now this is a sum of infinitely many terms and life is too short for us to keep adding terms forever. Instead we consider the sums of finite series caused by truncating the number of terms at some point well short of infinity! In fact, we know that if we stop after N terms, we find

$$1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^{N-1}} = \frac{1 - \frac{1}{2^N}}{1 - \frac{1}{2}} = 2 - \left(\frac{1}{2}\right)^{N-1}$$

Now whatever value we choose for N, this sum will be less than 2, but by choosing N large enough we can reduce the deficit to make it as small as we please. So we say that *if* we could add infinitely many terms the infinite sum makes sense if we assign to it the value 2. Clearly, no number less than 2 has the property that it can be approached in this manner, and equally clearly no number greater than 2 has this property either. The number 2 is unique in this respect and so we *define*

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2.$$

We have a new sort of sum, but it all makes perfect sense.

What Cauchy did was to apply this same logic to fractions. Take a standard example, one used by many writers, including Marx. Suppose we seek the slope of the parabola $y = x^2$ at the point $(X,\ X^2)$. We proceed by looking at the slope of a chord joining this point to a nearby point $\big(X + \Delta X,\ (X + \Delta X)^2\big)$. The slope of this chord is

$$\frac{(X + \Delta X)^2 - X^2}{(X + \Delta X) - X}$$

which simplifies to $2X + \Delta X$.

And now we say, in analogy with the series case discussed above: We can make the quotient approach as close as we like to the value $2X$ merely by suitably decreasing the size of ΔX. Note that we emphatically *do not* put $\Delta X = 0$. Had we done that, then the formula for the slope of the chord would have been reduced to $\frac{0}{0}$, which has no definite value.

Marx's manuscripts devote great attention to the problems caused by the way in which $\frac{0}{0}$ turns up in places where we don't want it. He wrote two long essays directly on this matter: "On the Concept of the Derived Function" and "On the Differential"; there are in addition several drafts of a proposed history of calculus, two versions of an unfinished essay on Taylor's Theorem, and various notes and addenda.

They all display a wide and deep acquaintance with the various attempts to address the logical difficulties, but there is one important omission: Cauchy. Now Cauchy's solution of the problem had been extant for 50 years by the date ascribed to Marx's work. I am inclined to the view that in spite of this, Marx was unaware of it, despite the fact that he had read widely on the subject and his accounts aim at comprehensiveness. Struik however takes a different view. According to him, there were, by the time Marx wrote, textbooks incorporating Cauchy's insights, and (on Struik's view) Marx would most likely have read them. He has a different take on the matter.

Struik believed that Marx ignored Cauchy's work because he saw nothing new in it. On this interpretation, Cauchy was essentially repeating an earlier approach. This one was that of Jean Le Rond d'Alembert (1717–1783). d'Alembert did not invoke the notion of a limit, but rather stressed the *order* in which the various parts of the calculation were performed. In our example, we can first simplify $\frac{(X+\Delta X)^2 - X^2}{(X+\Delta X) - X}$ to find $2X + \Delta X$ and *once this simplification is performed*, we may *then* set $\Delta X = 0$. The d'Alembert approach may be used to differentiate many standard functions, but it lacks the unifying notion supplied by the concept of the limit. There is no reason given for the need to simplify before setting $\Delta X = 0$.

What Marx did point out (although to my mind at tedious length) was that once one had a derivative for x^2, then the product rule could be pressed into service to supply derivatives for x^n for all values of n, and by extension to all polynomials. Others have further noted that we can go even further and use the quotient rule and so differentiate any rational function. (A rational function is the quotient of two polynomials.)

This however leaves us well short of a truly *general* method. In a further paper, Marx toyed with the idea that we could use Taylor series to supply derivatives. Recall that the Taylor series of a function $f(x)$ about $x = X$ is

$$f(x) = f(X) + f'(X)(x - X) + \frac{1}{2!}f''(X)(x - X)^2 + \ldots$$

This means that *if we already know the Taylor series* for $f(x)$, then we may read off the values of the derivatives (of all orders) from the series. So, for example, we can determine from the binomial theorem that

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{35x^5}{1280} - \dots$$

(As an exercise, find the pattern behind the coefficients.) So we can immediately read off the results: $f'(0) = \frac{1}{2}$, $f''(0) = -\frac{1}{4}$, etc. Or we can differentiate term by term and so find

$$f'(x) = \frac{1}{2} - \frac{x}{4} + \frac{3x^2}{16} - \frac{5x^3}{32} + \frac{7x^4}{1280} - \dots$$

which is the series for $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, the correct answer. However, this strikes me as a very clumsy approach.

The problem is that this method of defining derivatives puts the cart before the horse. The more usual order of procedure is to use a *prior* knowledge of the derivatives to reach the Taylor series. This point becomes critically important when we do not already know the series. Take as an example $f(x) = \sin x$.

Without using our knowledge of the derivatives, there is probably no way of deducing the series for this function. (I once believed that an old textbook, now extremely rare, had managed to do this, and spent long hours in trying to reconstruct how I thought it must have argued. Eventually I found myself in discussion with a retired mathematician who possessed a copy of this very text. True it did not make *explicit* use of calculus, but it gave an argument equivalent to the calculus approach.)

Now consider how we might proceed to find the derivative of $\sin x$. Follow the same method as used in the case of x^2. We need to consider the expression

$$\frac{\sin(X + \Delta X) - \sin X}{\Delta X}.$$

We can expand the numerator by means of the standard addition rule, and so find for the slope of the chord

$$\sin X \left(\frac{\cos \Delta X - 1}{\Delta X} \right) + \cos X \left(\frac{\sin \Delta X}{\Delta X} \right).$$

Now, if we attempt the d'Alembert approach, we run into the $\frac{0}{0}$ problem (twice!) and so are stymied.

However, the Cauchy limit attack on the problem works (after a few preliminaries). Here is one way to proceed. It may be established (e.g. on geometric grounds) that if $0 < \Delta X < \pi$, then $\sin \Delta X < \Delta X < \tan \Delta X$. (Remember that we are here working in *radians*.) This inequality implies $1 < \frac{\Delta X}{\sin \Delta X} < \sec \Delta X$, or equivalently $\cos \Delta X < \frac{\sin \Delta X}{\Delta X} < 1$. Now let ΔX shrink towards zero; $\cos \Delta X$ will tend upwards toward 1, and so the value of $\frac{\sin \Delta X}{\Delta X}$ is so-to-speak squeezed between a value that gets arbitrarily close to 1 and 1 itself. The limit of $\frac{\sin \Delta X}{\Delta X}$ as ΔX shrinks toward zero has to be 1. As was the case with the series example above, no other value will do. We write

$$\lim_{\Delta X \to 0} \frac{\sin \Delta X}{\Delta X} = 1.$$

Now look at the factor multiplying $\sin X$ in the expression for the slope of the chord. We can proceed as follows:

$$\begin{aligned}\frac{\cos\Delta X - 1}{\Delta X} &= -\frac{(1-\cos\Delta X)\,(1+\cos\Delta X)}{\Delta X\,(1+\cos\Delta X)} \\ &= -\frac{\sin^2\Delta X}{\Delta X\,(1+\cos\Delta X)} = -\left(\frac{\sin\Delta X}{\Delta X}\right)\left(\frac{\sin\Delta X}{1+\cos\Delta X}\right)\end{aligned}$$

and now we have the product of two factors. Of these, the first has just been discussed and found to have the limit 1, while the second is easily seen to have the limit 0. In the limit therefore this product is zero, and now we may use the information just discovered to deduce that

$$\lim_{\Delta X\to 0}\frac{\sin(X+\Delta X)-\sin X}{\Delta X} = (0\times\sin X) + (1\times\cos X) = \cos X.$$

So to my mind, Marx actually contributed very little to the debate over the foundations of calculus. His writing came too late; the key issue was already resolved, and furthermore his analyses are prolix and clumsy. They seem to beat about the bush and become repetitive while hardly advancing at all. The distinctions he makes between the various pre-Cauchy approaches are subtle, perhaps overly so, but ultimately beside the point.

Well that is my view. But it is only fair to point out that there are eminent mathematicians who see matters quite differently. Three in particular deserve mention here. Struik has already been noted. His original paper has since been reprinted and made more accessible. His article "Marx and Mathematics" was republished in 1997 in a book edited by Arthur Powell and Marilyn Frankenstein, *Ethnomathematics: Challenging Eurocentrism in Mathematics Education* , State University of New York Press, Albany, pp. 173–192. As I wrote above, it provides the best single introduction to this material, although I find it over-sympathetic to Marx.

Another good discussion is that in Joseph Dauben's "Marx, Mao and Mathematics: The Politics of Infinitesimals". (Dauben was a student of Struik's.) A version of this is posted on the web, and is most easily found by googling 'dauben marx mao' and asking for the html version. Sadly this version is rather badly corrupted. With a bit of practice and ingenuity, most of the actual text can be reconstructed, but the details of the references sadly can not.

Dauben notes a beneficial political effect of the discovery of this material. During the Cultural Revolution, Mao denounced the work of pure mathematicians as not contributing to the advancement of proletarian ideals. However, once it was revealed that the great Karl Marx had written on infinitesimals, then the work of Robinson was suddenly viewed in a different light. Pure mathematicians were instantly OK after all, and persecution of them ceased.

The third mathematician to take Marx's mathematical work seriously is Paulus Gerdes. Gerdes is an African mathematician (based in Mozambique) with a long-standing interest in raising the profile of African mathematics. He is prominent in the movement known as *Ethnomathematics* that celebrates the achievements of cultures outside the Euro-American mainstream.[2] His book *Marx Demystifies Mathematics* was originally written in Portuguese, but has been translated into English. It has been widely praised but also has not been without its critics. One particularly trenchant review was posted on the Amazon website; its author is the mathematician Jacob Kesinger. He writes: "The title is a bit misleading; this book represents Marx's efforts to put calculus on a sound, rigorous footing. As a mathematician, I have to say that Marx succeeds only in moving the handwaving from one area to another. If the author was not a mathematician, he should have made an attempt to familiarize himself with the actual rigorization of calculus in the nineteenth century (in, for example, the work of Cauchy). If the author was a mathematician, he most certainly should have known better. I cannot recommend this book to anyone who does not have a solid understanding of mathematics."

The interesting and unanswered question behind all this is: Why did Marx take such an interest in Mathematics, and in particular, these somewhat esoteric aspects of it? Here most of the sources are either silent or else so impenetrable that it is quite unclear what they are trying to say. Struik believes that it was simply a private interest. However other commentators have tried to connect this interest to Marx's main work.

The principal intellectual influence on Marx was the philosophy of Georg Hegel (1770–1831). Hegel is not an easy philosopher to understand. His best-known contribution to subsequent thought is the so-called 'Hegelian dialectic'. The word 'dialectic' relates to the idea of *dialogue*, and Hegel is concerned to address the question of *real* dialogue between speakers who disagree about some point or other. It is not the same as logic. Logic, in an extremely limited field, can of course ensure correct conclusions from agreed postulates. However, in the real world things are rather more messy. Hegel thought that when two persons set out to discuss some question or other, neither will actually be completely right or wrong. Rather Person A will enunciate some position (the "thesis"), which will be challenged by Person B who takes a different view (the "antithesis"). As neither is completely right or completely wrong, it is open to the two disputants to agree on some synthesis that combines the best features of both initial viewpoints.

Hegel has been described as an *idealist*, one who sees the world as composed of ideas rather than of things. However this simple description should not be taken quite at face value. Idealists do not, of course, deny the existence of an external world composed of *things*; they do however make the strong point that whenever we seek to analyze or to discuss this external world we necessarily rely on our *ideas* about it.

This is where Marx took issue with Hegel. Marx was a *realist*. In other words, he placed paramount emphasis on what was *really* going on in the external world. Marx's 'dialectical materialism' became the official ideology of the socialist (commu-

[2] Ethnomathematics is currently much in vogue. It has a ring of political correctness about it, that suits the contemporary climate of thought. However, many claims of the ethnomaticians are somewhat grandiose. I published a critique in my *Function* column for June 1997. The alert reader will note that the reprinted version of Struik's paper was published in an ethnomathematical collection. However, the editors were at some pains to justify its inclusion. Struik was not really writing in this vein.

nist) nations for a period of about three-quarters of a century. Of his relation to Hegel's thought Marx had this to say:

"My dialectic method is not only different from the Hegelian, but is its direct opposite. To Hegel, the life-process of the human brain, i.e., the process of thinking, which, under the name of 'the Idea', he even transforms into an independent subject, is the demiurgos of the real world, and the real world is only the external, phenomenal form of 'the Idea'. With me, on the contrary, the ideal is nothing else than the material world reflected by the human mind, and translated into forms of thought." (From his book *Capital [Das Kapital]*, Volume 1, p. 29).

But where Marx *did* agree with Hegel was probably more important. Both believed that the processes of human history followed the *thesis-antithesis-synthesis* pattern of human dialogue. So the rise of capitalism (the "thesis") would lead inevitably to the impoverishment of the working classes, and this in turn would cause the workers to rise up and throw off the capitalist yoke, so giving place to the 'dictatorship of the proletariat', and the establishment of socialist societies (the "antithesis"). In due course, the state itself would wither away, and be replaced by an new and benevolent order: Communism (the synthesis).

[Looking back on things, we might say that the first of these processes did indeed actually occur, certainly in Russia, China and elsewhere, but there is no evidence *anywhere* of the subsequent development of true benevolent Communism!]

But what has all this got to do with mathematics?

Well, probably nothing, and reputable commentators like Struik make no attempt to connect dialectical materialism with Marx's mathematics. However this has not stopped others from trying to bridge this gap. Among 'true believers' in Marxism, there is a confusion between dialectic and logic, so that the view is expressed that the Hegelian dialectic adopted by Marx is a superior form of logic, and so of necessity it must result in a better form of mathematics than the traditional one that relies on formal logic only. Such a position informs much of the subsidiary material included in the books collecting Marx's mathematical papers, and somewhat more accessibly it underlies a glowing review of the English edition by the Marxist commentator Andy Blunden. This is posted on the web at

http://www.marxists.org/archive/marx/works/1881/mathematical-manuscripts/review.htm

This review glows with triumph over the perceived superiority of Marx's version of calculus over 'capitalist mathematics', but it has nothing at all to say about the mathematics itself, and makes no attempt to demonstrate where this superiority actually lies!

All in all, I regard Marx's contributions to mathematics as negligible, although it is of interest that they exist and are now available. However, if some other less famous author had produced them, no-one would have taken the slightest notice!