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The Complex Roots of a Quadratic Equation: A Visualization

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Introduction

Visualization is a powerful tool in any teacher's bag of tricks, perhaps even more so for the mathematics teacher who must often visually present abstract concepts. At times, though, we may be at a loss as to how to visually represent a particular notion. For example, when the roots of a quadratic equation are real numbers, the graph of the corresponding quadratic function intersects the horizontal axis in two points (or just one point if the root is of multiplicity 2). See Figure 1. In this case, the student can easily see the relationship between the roots and the graph of the function along with other concepts such as the symmetry of the roots about the axis of the parabola. However when the roots are complex the graph does not intersect the x-axis, as shown in Figure 2, so how do we 'see' them in this case?

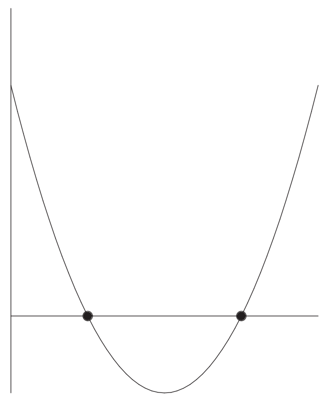

Figure 1: Real roots

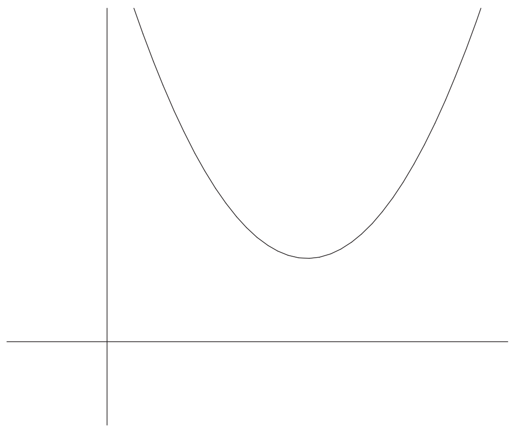

Figure 2: Complex roots. Where are they?

Over the past twenty-five or thirty years several methods for visualizing the complex roots of a quadratic equation have appeared in the literature (see the references for a sample of articles). With the advent of technological tools for graphing, these methods deserve to be more widely known especially among secondary school teachers. My intent here is to demonstrate several of these methods. Before doing so, however, I want to present some preliminaries regarding quadratic equations, complex numbers, and plotting in space.

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Preliminaries

1. Our primary concern will be the quadratic equation,

$$az^2 + bz + c = 0$$

and the corresponding quadratic function, $p(z) = az^2+bz+c$, in which we take a, b, and c to be real numbers with $a > 0$. There is no loss of generality in this last assumption; all our parabolas will be drawn concave upward. We will also use the *quadratic formula*,

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Lastly we note that the abscissa of the vertex of the parabola is $-\frac{b}{2a}$ and that its axis of symmetry is $x = -\frac{b}{2a}$.

2. A complex number z may be written as $z = x+iy$ where x and y are real numbers and $i = \sqrt{-1}$. The number x is called the *real part* of z and y is its *imaginary part*. We will presume a familiarity with the basic arithmetic of complex numbers. Because we can associate the complex number $z = x + iy$ with the ordered pair (x, y), we may depict the complex numbers as points in the Cartesian plane. When depicted in this way, the x-axis is called the real axis, the y-axis is called the imaginary axis, and the plane is referred to as the *complex plane*, Figure 3.

The *conjugate* of a complex number is the number $\overline{z} = x - iy$. This is important because we know that for a polynomial with real coefficients, its zeros come in conjugate pairs. This is a rather easy statement to prove using the facts $\overline{zw} = \overline{z} \cdot \overline{w}$, $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{\overline{z}} = z$, and the fact that z is real iff $\overline{z} = z$. Graphically, $\overline{z}$ is the reflection of z in the real axis. The absolute value or *modulus* of a complex number $z = x + iy$ is defined to be the real number $|z| = \sqrt{x^2 + y^2}$. It is a generalization of the notion of absolute value for real numbers because if z is real, then $y = 0$ so $|z| = \sqrt{x^2 + 0^2} = \sqrt{x^2} = |x|$. Geometrically, the modulus of z is the distance between z and the origin in the complex plane. It is always true that $|z| \geq 0$. Figure 4 shows the modulus and conjugate of z.

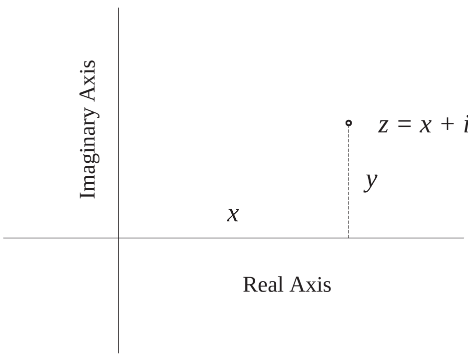

Figure 3: The complex plane

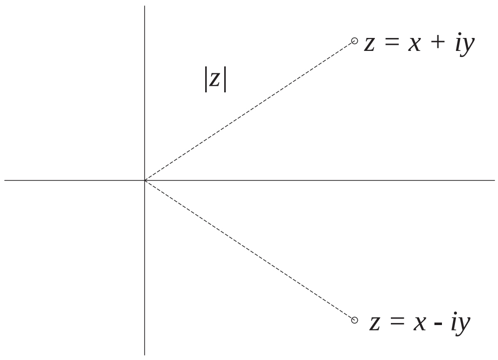

Figure 4: Modulus and conjugate

3. To plot the graph of a function $w = p(z)$ we need ordered pairs (z, w) and when $z = x+iy$ and $w = u+iv$ are complex numbers, these ordered pairs give rise to ordered

4-tuples (x, y, u, v) of real numbers. Such a graph would have to be drawn using four mutually perpendicular axes; that is, in Cartesian 4-dimensional space. Since this is not possible, we shall resort to a method that requires only ordered triples (x, y, u) of real numbers. Thus, we will be plotting *surfaces* in three-dimensional space using three mutually perpendicular lines as axes, labeled x, y, and u as shown in Figure 5. The x and y-axes determine the complex plane as we indicated before and, for the first method we shall present, the modulus of a complex number will be plotted along the u-axis.

For the first two methods, we assume that z is complex and for third method, we will take z to be real.

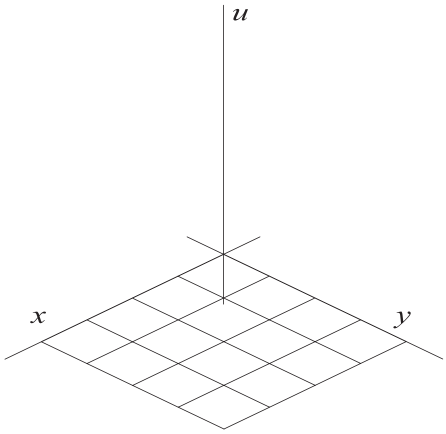

Figure 5: 3D Axes

Method 1. The Modulus Surface

The graph of the modulus of $p(z)$ is admirably suited to depicting its zeros whether they are real or complex; moreover, this graph is reasonably easy to draw with the right technological tools. Since the modulus of $p(z)$, $u = |p(z)|$, is a real number, its graph is just a surface in space called the *modulus surface* of $p(z)$. We note that since $|p(z)| \geq 0$, the graph always lies on or above the complex plane. In addition, $|p(z)| = 0$ if and only if $p(z) = 0$ so the graph is incident with the plane only at those values of z for which $p(z) = 0$; that is, only at the zeros of $w = p(z)$. Figures 6 and 7 show the modulus surfaces of $p(z) = z^2 - 4z + 3 = (z - 1)(z - 3)$ and $p(z) = z^2 - 4z + 8$. The first has real roots at $z = 1, 3$ while the second has complex roots at $z = 2 + 2i$ and $z = 2 - 2i$. In each case, the surface touches the xy-plane at exactly these points as is easily seen.

One of the nice features about the modulus surface is that it can be used with quadratic polynomials whose coefficients are not necessarily real; that is, it works in all cases. If you have the capability of graphing such surfaces, try plotting the modulus surface of a quadratic with a repeated root such as $z^2 - 4z + 4$ or one with complex coefficients such as $z^2 - 2iz + 1$. Another nice feature of the modulus surface is that

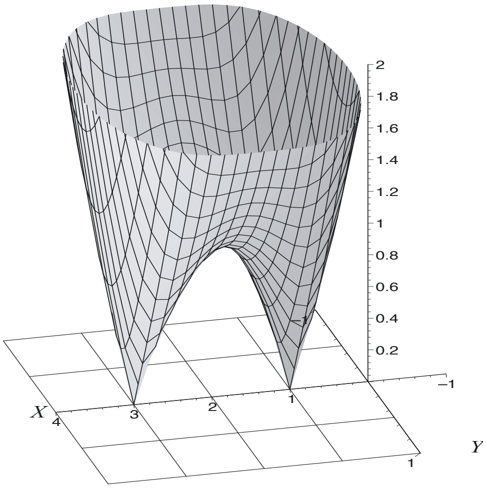

Figure 6: Modulus surface of
$p(z) = z^2 - 4z + 3$.
Real roots at 1 and 3.

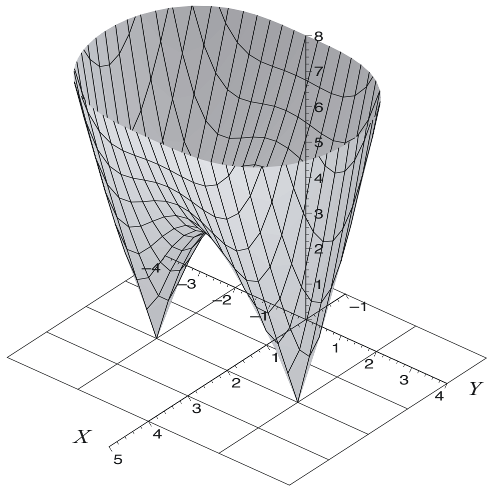

Figure 7: Modulus surface of
$p(z) = z^2 - 4z + 8$.
Complex roots at $2 + 2i$ and $2 - 2i$.

it also works to exhibit the roots of polynomials of higher degree. Try, for example, plotting various cubic and quartic polynomials.

Method 2. The Real and Imaginary Parts

As mentioned in the Introduction, when the roots of a quadratic equation $ax^2 + bx + c = 0$ are complex, the corresponding quadratic function $p(x)$ has a graph similar to that in Figure 2. But this figure only presents a narrow view of the entire graph. This occurs because we restrict the values of x to the real numbers so that $p(x)$ will be real also. But $p(x)$ is real valued when x is restricted to other values as well. An examination of all the values of the independent variable for which p is real will give us the bigger picture and a graph showing the complex roots.

We split $p(z)$ into its real and imaginary parts and require $p(z)$ to be real. Each of the two parts will be quadratic and plotted as curves (parabolas) in space rather than curves in a plane. We will also show a connection to the previous method. We begin with a some algebra.

Let $u = p(z) = az^2 + bz + c$ where $z = x + iy$. After substituting $x + iy$ for z and gathering real and imaginary parts, we obtain

$$p(z) = (ax^2 - ay^2 + bx + c) + (2axy + by)i$$

Requiring $p(z)$ to be real means that we must have

$$2axy + by = y(2ax + b) = 0$$

. Therefore, $y = 0$ or $x = -\frac{b}{2a}$. Hence $p(z)$ is real when z is on the line $y = 0$, $u = 0$ or on the line $x = -\frac{b}{2a}$, $u = 0$; both lines lie in the complex plane. When z is restricted to the line $y = 0$, $u = 0$; that is, when restricted to the x-axis, $z = x$ and $p(z)$ may be written as $p(x) = ax^2 + bx + c$. When the zeros of $p(x)$ are complex, its graph does not intersect the

x-axis. We refer to this part as the *real part* of $p(z)$. See Figure 8 for an example. When z is restricted to the line, $x = -\frac{b}{2a}$, $u = 0$; that is the line $z = -\frac{b}{2a} + yi$ is parallel to the y-axis, then $p(z)$ becomes

$$p(-\frac{b}{2a} + yi) = \left(a\frac{b^2}{4a^2} - ay^2 - \frac{b^2}{2a} + c\right) + \left(2a\frac{-b}{2a}y + by\right) i$$

or

$$p(-\frac{b}{2a} + yi) = -ay^2 + \frac{4ac - b^2}{4a}$$

which we shall denote by $q(y)$. That is,

$$q(y) = -ay^2 + \frac{4ac - b^2}{4a}.$$

We will call $q(y)$ the *imaginary part* of $p(z)$. In this case, the graph of q *does* intersect the line $z = -\frac{b}{2a} + yi$ and the zeros of $p(z)$ appear at these points of intersection. See Figure 9. Note that both parabolas share a common vertex, $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$ as shown in Figure 10 (and Figure 11). The points of intersection of $q(y)$ with the complex plane in these two figures are precisely at the (complex) zeros of $p(z)$.

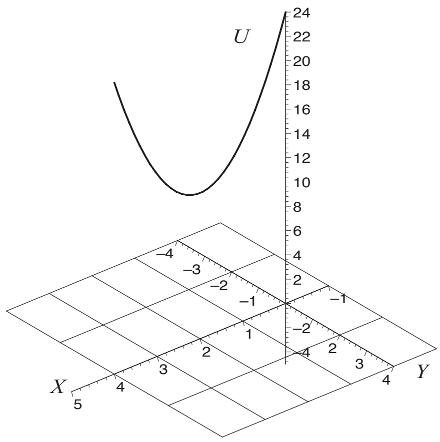

Figure 8: Real branch of $p(z) = 3(z^2 - 4z + 8)$

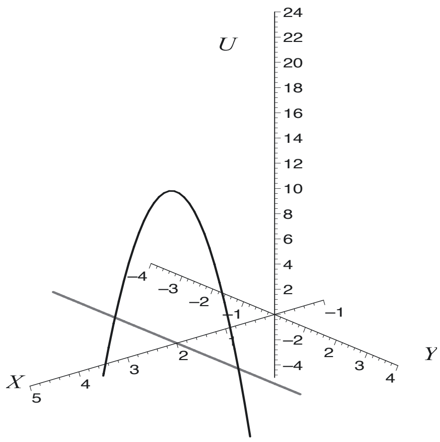

Figure 9: Imaginary branch of $p(z) = 3(z^2 - 4z + 8)$

Note the relationship between this method and that of the modulus surface. These graphs are just traces of the modulus surface. The real part of $p(z)$ is obtained by slicing it with the plane $y = 0$. Can you determine how to obtain the imaginary part of $p(z)$? See Figures 12, 13, and 14.

Method 3. The Common Case

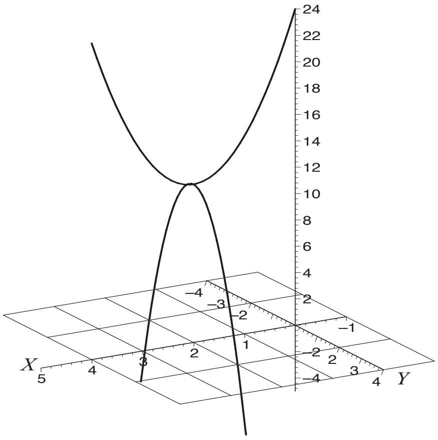

Figure 10: The two branches of $3(z^2 - 4z + 8)$

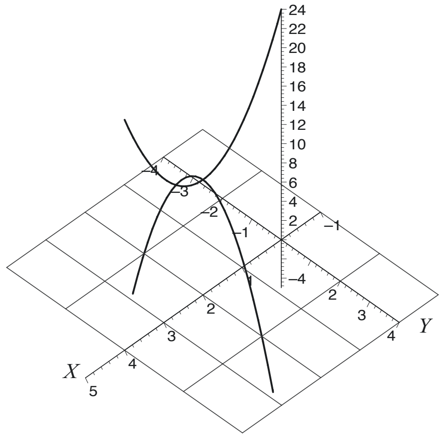

Figure 11: Another view; zeros at $2 \pm 2i$

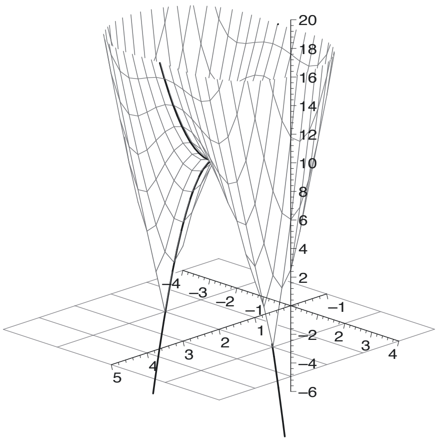

Figure 12: Traces

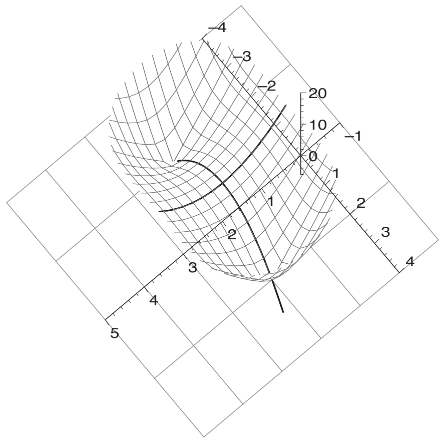

Figure 13: An inside view

This method is called the common case because it is the one we might commonly meet in a classroom situation—a quadratic equation with nonreal roots whose corresponding function has a graph like that shown in Figure 2. Here, we treat z as a real number and write our quadratic as $p(x) = ax^2 + bx + c$ with $a > 0$. Let $r_1 = -\dfrac{b}{2a} - \dfrac{\sqrt{b^2 - 4ac}}{2a}$ and $r_2 = -\dfrac{b}{2a} + \dfrac{\sqrt{b^2 - 4ac}}{2a}$. Since $p\left(-\dfrac{b}{2a}\right) = \dfrac{4ac - b^2}{4a}$, the vertex of the parabola is at

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right).$$

If we let $\alpha = -\dfrac{b}{2a}$ and $\beta = \dfrac{\sqrt{b^2 - 4ac}}{2a}$, then

$$r_1 = \alpha - \beta,\ r_2 = \alpha + \beta$$

and the vertex is at $(\alpha, -a\beta^2)$. Note that the zeros of $p(x)$, when real, lie on a chord

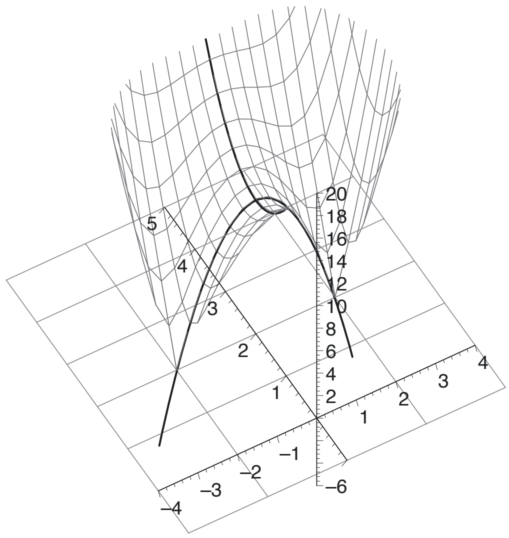

Figure 14: A view from beneath the surface

parallel to the x-axis that is $a\beta^2$ units away from its vertex. See Figure 15.

When the zeros of $p(x)$ are complex numbers, write

$$r_1 = -\frac{b}{2a} - i\frac{\sqrt{4ac-b^2}}{2a},$$

$$r_2 = -\frac{b}{2a} + i\frac{\sqrt{4ac-b^2}}{2a}.$$

How might we 'see' these zeros graphically? Obviously, we cannot but we can locate two points on the graph of $p(x)$ that are associated with them. As before, let $\alpha = -\frac{b}{2a}$ but take $\beta' = \frac{\sqrt{4ac-b^2}}{2a}$. Then $p(\alpha) = a\beta'^2$ so that the vertex of the parabola is now at $(\alpha, a\beta'^2)$. Draw a chord $2\beta'$ units long parallel to the x-axis and $2a\beta'^2$ units above it. This chord will intersect the parabola in two points, s_1 and s_2. It is easy to see that $s_1 = (\alpha - \beta', 2a\beta'^2)$ while $s_2 = (\alpha + \beta', 2a\beta'^2)$. Thus, *the first coordinates of s_1 and s_2 are respectively the difference and sum of the real and imaginary parts of the zeros of $p(x)$* and are the two points we seek. See Figure 16.

There are several ways to easily locate these points.

(a) The Auxiliary Polynomial

This polynomial is defined by the equation,

$$q(x) = -p(x) + 2p\left(-\frac{b}{2a}\right).$$

$q(x)$ and $p(x)$ share the same vertex but $q(x)$ has opposite concavity. Because of this, if $p(x)$ has complex zeros, $q(x)$ will have real zeros (and vice versa). Now plot $p(x)$ and $q(x)$ on the same axes, then draw two lines perpendicular to the x-axis passing through

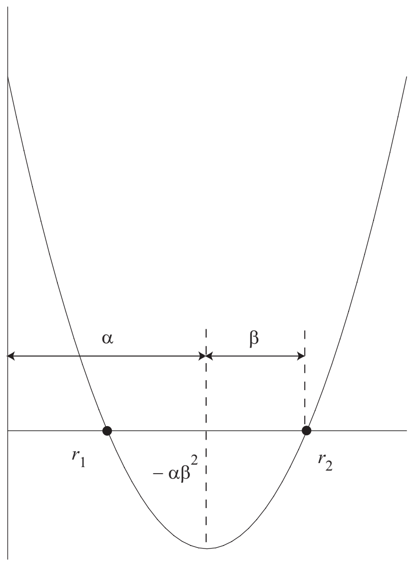

Figure 15: The real zeros

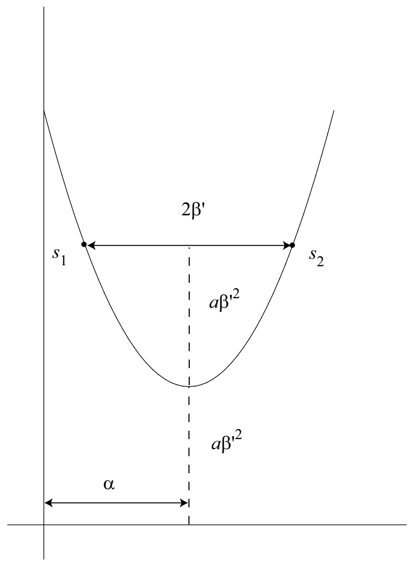

Figure 16: The complex case

the zeros of $q(x)$. These lines will intersect $p(x)$ at the points s_1 and s_2. To verify this statement, note that the zeros of $q(x)$ are $-\dfrac{b}{2a} \pm \dfrac{\sqrt{4ac-b^2}}{2a} = \alpha \pm \beta'$. It is then easy to see that $q(\alpha \pm \beta') = 0$ and $p(\alpha \pm \beta') = 2a\beta'^2$. Figure 17 shows $p(x)$ and its auxiliary polynomial together with the vertical lines mentioned above.

Example: $p(x) = x^2 - 6x + 13$

In this case, $q(x) = -x^2 + 6x - 5 = -(x-1)(x-5)$ so the zeros of $q(x)$ are $r_1 = 1$ and $r_2 = 5$. Note also that $\alpha = 3$ and $\beta' = 2$. We therefore determine the zeros of $p(x)$ to be $\alpha \pm \beta' i = 3 \pm 2i$. These are easily verified using the quadratic formula. If we draw the graphs of $p(x)$ and $q(x)$ on the same coordinate axes as previously described, we may easily locate the points on $p(x)$ associated with these complex zeros. In this case, they are $(1, 8)$ and $(5, 8)$. Thus $\alpha - \beta' = 1$ and $\alpha + \beta' = 5$. Hence, we have Real-Part-of-the-zeros $= \dfrac{(\alpha+\beta')+(\alpha-\beta')}{2} = 3$ and Imaginary-Part-of-the-zeros $= \dfrac{(\alpha+\beta')-(\alpha-\beta')}{2} = 2$ so the zeros are at $3 \pm 2i$ as obtained above. See Figure 18.

(b) The Tangent Line Method

If $p(x) = ax^2 + bx + c$, then its derivative is $p'(x) = 2ax + b$. Since the coordinates of s_2, say, are $(\alpha+\beta', 2a\beta'^2)$, we note that $p'(\alpha+\beta') = 2a(\alpha+\beta')+b = 2a\dfrac{\sqrt{4ac-b^2}}{2a} = 2a\beta'$. Thus the slope of the tangent line to the parabola at s_2 is $2a\beta'$ and the line is easily seen to be

$$y_2 = 2a\beta' x - 2a\alpha\beta'.$$

If $x = \alpha$, then $y = 0$ so this line contains the point $(\alpha, 0)$; note that α is the abscissa of the vertex of $p(x)$ so that the point $(\alpha, 0)$ lies on the axis of symmetry of $p(x)$ and below

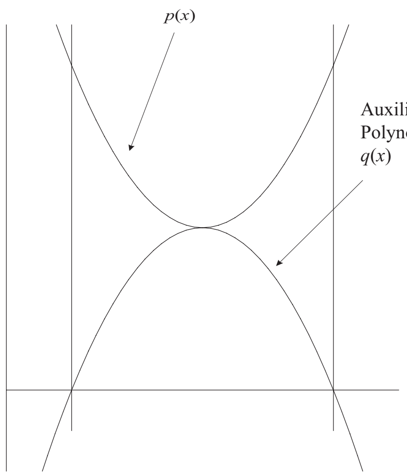

Figure 17: The auxiliarypolynomial

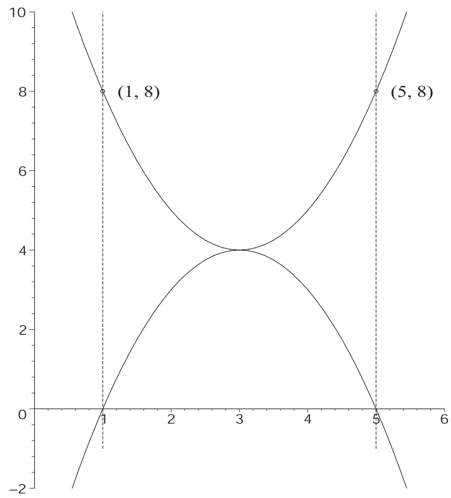

Figure 18: Locating the complex zeros of $p(x) = x^2 - 6x + 13$

its vertex. In the same way, the line,

$$y_1 = -2a\beta' x + 2a\alpha\beta'$$

is tangent to the parabola at the point s_1 and also contains the point $(\alpha, 0)$. Thus, if we place a straightedge on the point $(\alpha, 0)$ and parallel to the y-axis, then rotate it clockwise until it is tangent to the parabola, we arrive at the point s_2. Rotating the straightedge counterclockwise will locate the point s_1. See Figure 19.

Example: $p(x) = x^2 - 6x + 13$

We saw previously that the zeros of $p(x)$ are $3 \pm 2i$. Here we have $p'(x) = 2x - 6$, $\alpha = 3$, and $\beta' = 2$. Thus $p'(\alpha + \beta') = 4$ so that the line $y_2 = 4x - 12$. In the same manner, the line $y_1 = -4x + 12$.

Interconnections

In an attempt to tie these three methods together, we present several plots of a quadratic polynomial using different points of view. To begin with, Figure 20 shows a graph of a quadratic polynomial with complex zeros.

Now suppose that this is a 3D-plot and that our perspective is that the horizontal axis in this figure is the x-axis while the vertical axis is the u-axis and imagine that the y-axis is perpendicular to the plane of the paper. If we then rotate the graph, we get a view of the plot as shown in Figure 21 and we see that our graph is the graph of the real part of the quadratic as described in Method 2.

Figure 22 shows the plot in Figure 20 with its auxiliary polynomial added and Figure 23 shows that same plot rotated in space giving us a viewpoint like that in Figure 21.

In Figure 24, we have added the imaginary part of the quadratic as described in the second method. Note that the auxiliary polynomial and the imaginary part intersect

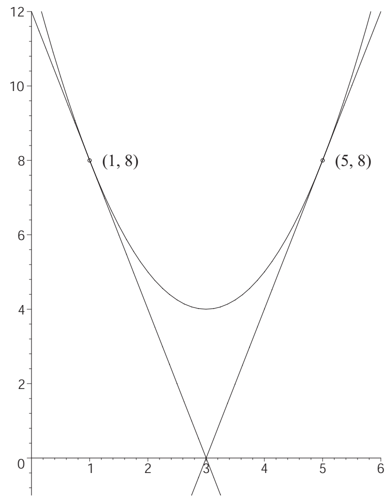

Figure 19: Locating the complex zeros of $p(x) = x^2 - 6x + 13$, tangent line method

the complex plane in four points. These four points lie on a circle whose center and radius are easily determined. Using the notation above, this circle is $|z - \alpha| = |\beta'|$. Figure 25 shows this circle. Lastly, Figure 26 shows these three parabolas together with the modulus surface of $p(z)$.

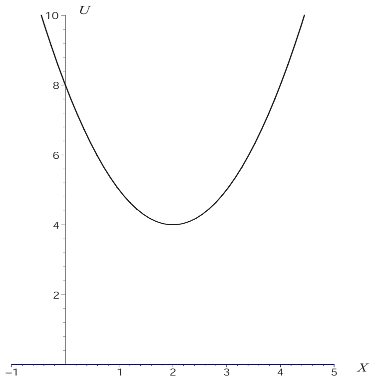

Figure 20: Quadratic polynomial complex zeros

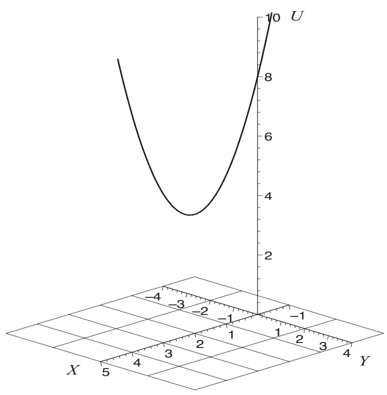

Figure 21: 3D view.

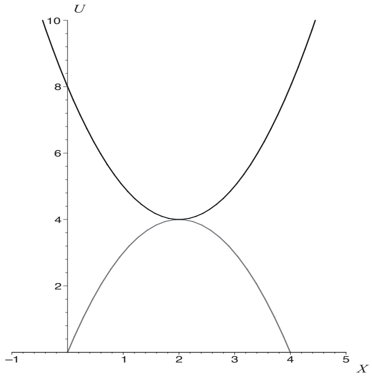

Figure 22: Quadratic polynomial and auxiliary polynomial with complex zeros

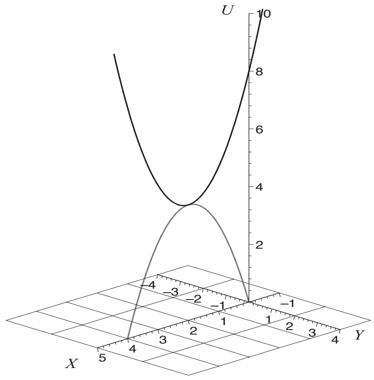

Figure 23: 3D view.

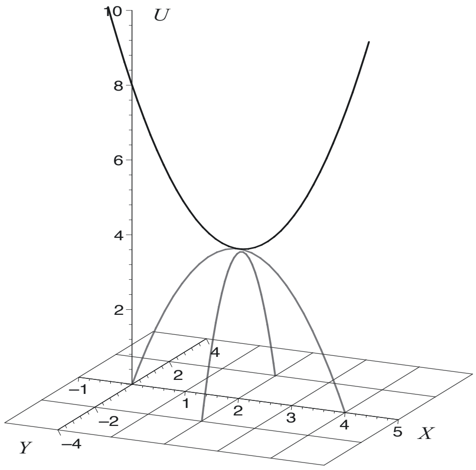

Figure 24:

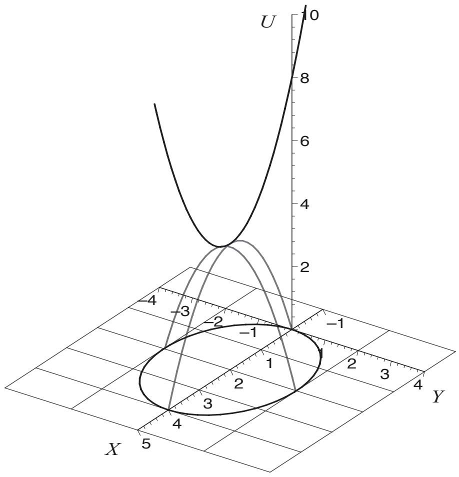

Figure 25:

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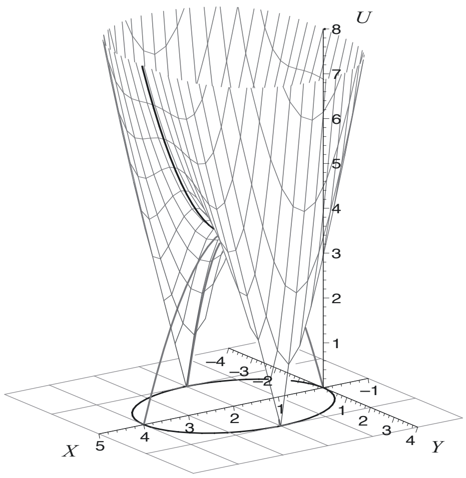

Figure 26: