Problems 1311–1320

Q1311   Prove that $\tan 75° - \tan 60° = 2$ using purely geometrical arguments.

Q1312   Three right-angled triangles have integral side lengths. The side lengths of the first and second triangles are, respectively, 5, 12, 13, and 8, 15, 17 (units). Given that the length of the hypotenuse of the third triangle is 221 units, find the other two lengths.

Q1313   This year (2009), November 13th falls on Friday. Explain why every year must have at least one Friday the thirteenth. What is the largest number of these days that can fall in a year?

Q1314   Prove that in an acute-angled triangle ABC there holds

$$\tan^n A + \tan^n B + \tan^n C \geq 3 + \frac{3n}{2},$$

where $n = 0, 1, 2, 3, \ldots$.

Q1315   Let a and b be two integers. Prove that $2a + 3b$ is divisible by 17 if and only if $9a + 5b$ is divisible by 17.

Q1316   The area S and the angle γ opposite side c of a triangle are given. Determine the other two sides a and b such that c is minimum.

Q1317   Given a_1, a_2, a_3, a_4, a_5 satisfying

$$a_1^2 + \cdots + a_5^2 = 1,$$

prove that

$$\min_{1 \leq i \neq j \leq 5} (a_i - a_j)^2 \leq \frac{1}{10}.$$

Q1318   Find all primes a, b, and c satisfying

$$abc < ab + bc + ca.$$

Q1319   Find all functions f satisfying

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}.$$

Q1320   Find all functions f satisfying

$$f\left(\frac{x+y}{2}\right) = \frac{2f(x)f(y)}{f(x) + f(y)}.$$