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"Now I know": Solving logical puzzles using graphs

Catherine Greenhill[1]

It's not every day that a mathematics puzzle makes it into mainstream media. But that's what happened recently with "Cheryl's Birthday problem". This problem was posted by Kenneth Kong, the host of a Singaporean TV show, on his Facebook page [1] on 10 April, and it went viral. Since then, the problem has appeared in such esteemed publications as the *New York Times* [2], the BBC [3] and the *Sydney Morning Herald* [4]. The question is given below, lightly reworded for clarity.

Cheryl's Birthday:
Albert and Bernard just became friends with Cheryl, and they want to know when her birthday is. Cheryl gives them a list of 10 possible dates:

May 15	May 16	May 19
June 17	June 18	
July 14	July 16	
August 14	August 15	August 17

Cheryl then tells Albert the month of her birthday, and separately tells Bernard the day of her birthday. Then Albert and Bernard have this conversation:

Albert:	I don't know when Cheryl's birthday is, but I know that Bernard does not know either.
Bernard:	At first I didn't know when Cheryl's birthday is, but now I know.
Albert:	Now I know too.

So when is Cheryl's birthday?

A similar problem was posted a couple of days later by Tanya Khovanova on her blog [5] (again, I have done some very mild rewording, for clarity).

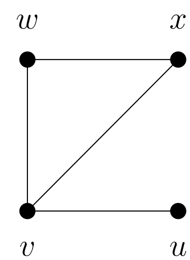
Tanya's Number:
Tanya thought of a positive integer that is below 100 and is divisible by 7. In addition to the public knowledge above, Tanya privately told the units digit of her number to Alice and she told the tens digit to Bob. Alice and Bob are very logical people, but their conversation might seem strange:

Alice:	You do not know Tanya's number.
Bob:	Now I know Tanya's number.

[1] A/Prof Catherine Greenhill is an Associate Professor in the School of Mathematics and Statistics at UNSW Australia.

You might like to think about these problems yourself before reading any further.

Now I am a graph theorist, and so to me everything looks like a graph. I don't mean the graph of a function with an x-axis and y-axis, and I don't mean a bar graph (bar chart). I mean the kind of graph that has vertices (also called nodes), with some pairs of vertices being joined by edges (also called links). An example of a graph is shown below.



A lot of scientists use graphs as a model of their favourite complex system. For example, a physicist might form a graph where the vertices represent atoms and the edges represent bonds between the atoms. A social scientist might form a graph where the vertices represent people and the edges represent some kind of relationship between the people, such as friendship.

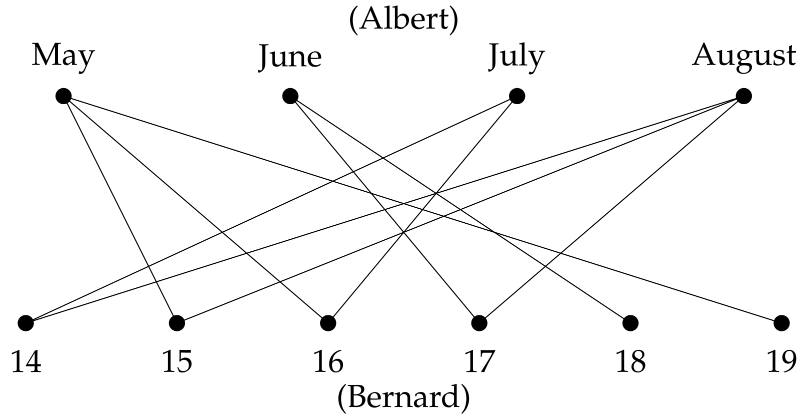
We will need a special kind of graph, called a *bipartite graph*, to help us solve these logic puzzles. In a bipartite graph the vertices are split into two groups, and an edge can only go *between* the groups. (The word "bipartite" means "two parts".) The graph given above is not bipartite. (Can you see why?)

We will also need a bit of terminology. The *degree* of a vertex is the number of edges which meet that vertex. In the example above, vertices w and x have degree 2, vertex v has degree 3 and vertex u has degree 1. We say that two vertices are *neighbours* if they are joined by an edge. So in the above example, w and x are neighbours but w and u are not.

For Cheryl's Birthday problem, one group of vertices will be labelled by the possible months (May, June, July, August) and the other group of vertices will be labelled by the possible days (14, 15, 16, 17, 18, 19). We draw an edge between a month and a date if that combination is one of the 10 possible birthdays on Cheryl's list. This leads to the bipartite graph shown on the next page.

Solving Cheryl's Birthday puzzle

Now let us solve Cheryl's Birthday problem, with the help of this graph. Albert and Bernard can both see the entire graph. Albert knows the correct vertex a in {May, June, July, August}, while Bernard knows the correct vertex in b in {14, 15, 16, 17, 18, 19}.



- Albert does not know when Cheryl's birthday is: this means that the degree of Albert's vertex a is at least 2. But this is true of all the "month" vertices: so we can deduce this from the graph.

- However, Albert knows that Bernard *does not know* Cheryl's birthday. This means that all of a's neighbours have degree (at least) 2. This rules out the months of May and June, since May has neighbour 19, and 19 has degree 1 (and similarly June has neighbour 18, which has degree 1). So we know that a is either July or August.

- Bernard says that at first, he did not know what Cheryl's birthday was. This means that his vertex b is not 18 or 19, as these vertices have degree 1 in the graph.

- But now that Albert has made his first statement, Bernard says that he *now knows Cheryl's birthday*. In other words, Bernard has followed the same logic as we have, and now knows that a is either July or August. Since he is now *certain* of Cheryl's birthday, it follows that b is neighbours with *precisely one* of July or August.[2] Hence b cannot be 14, as 14 is neighbours with both July and August. Therefore b is either 15, 16 or 17.

- Finally, Albert says that he now also knows Cheryl's birthday. Again, Albert has followed the same logic as us up until this point, so he knows that b is either 15, 16 or 17. For Albert to be *certain* now, it must be that a has precisely one neighbour in this list. But August has both 15 and 17 as neighbours, so a cannot be August.

- Therefore a must equal July, which in turn implies that b equals the unique neighbour of July in $15, 16, 17$, namely $b = 16$.

Hence Cheryl's birthday is July 16.

[2]Why? If b was not the neighbour of July or August then the problem would have no solution. If b was the neighbour of both July and August then Bernard cannot be certain of anything at this point.

The graph helps me to visualise the data and to make the logical deductions which are necessary to solve the puzzle. I hope you found the graph useful too.

Now we've worked through Cheryl's Birthday Puzzle, you might like to try this approach on the other puzzle, Tanya's number. The first step is to design the appropriate graph to hold all the data. Once you have had a go yourself, read on...

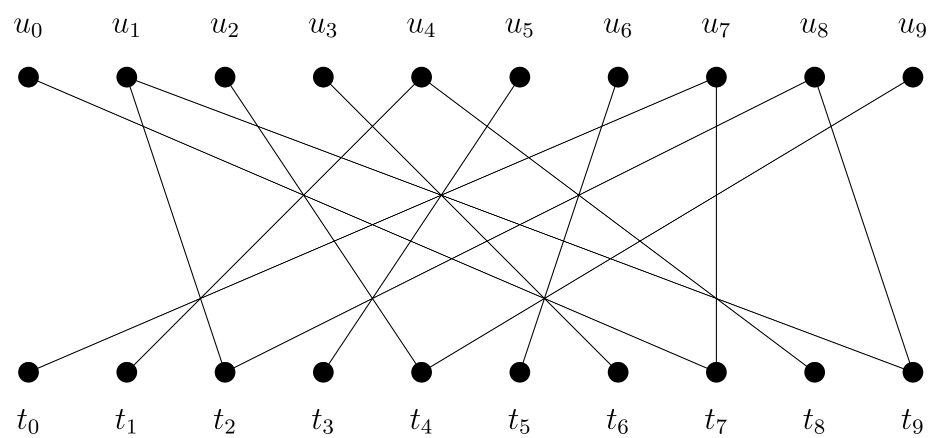
Solving Tanya's Number puzzle

For Tanya's Number puzzle, the bipartite graph has vertices representing the unit digit of the number on one side, and vertices representing the tens digit on the other. To avoid confusion, we will label the "units" vertices by $u_0, u_1, \ldots, u_9$ and we will label the "tens" vertices by $t_0, t_1, \ldots, t_9$.

The positive integers which are less than 100 and divisible by 7 are

$$7, \quad 14, \quad 21, \quad 28, \quad 35, \quad 42, \quad 49, \quad 56, \quad 63, \quad 70, \quad 77, \quad 84, \quad 91, \quad 98.$$

These 14 numbers will each be represented by an edge in the following graph. For example, the number 21 is represented by an edge between t_2 and u_1, while the number 7 is represented by an edge between t_0 and u_7.



Alice knows the correct "units" vertex a, and Bob knows the correct "tens" digit b.

- Alice says "You do not know Tanya's number". This means that she is *certain* that the degree of Bob's vertex b is at least 2. Hence *every* neighbour of Alice's vertex a must have degree at least 2. This implies that a cannot be u_3, u_4, u_5, u_6, u_7, since each of these vertices have at least one neighbour on the "tens" side with degree 1. (For example, u_3 has a unique neighbour, namely t_6, and t_6 has degree 1, while u_4 has two neighbours, namely t_1 and t_8, both of which have degree 1.)

- Now Bob has followed the same logic as us, so he knows that a must be one of u_0, u_1, u_2, u_8, u_9. Given this information, he now says that *he knows Tanya's number*. This mean that Bob's vertex b must be neighbours with *precisely one of* the vertices u_0, u_1, u_2, u_8, u_9.
  - The only vertices on the "tens" side which have a neighbour in this list are t_2, t_4, t_7 and t_9. (All the other "tens" vertices have a unique neighbour which does not belong to this list.) So b must be one of t_2, t_4, t_7 and t_9.
  - But t_2 is neighbours with both u_1 and u_8, and similarly for t_9. Therefore b cannot be t_2 or t_9 (or Bob would still be uncertain about Tanya's number). Similarly, t_4 is neighbours with both u_2 and u_9. So b cannot be t_4.
  - The only remaining possibility is that b is t_7, and then a must equal the unique neighbour of t_7 in the list u_0, u_1, u_2, u_8, u_9, namely 0.

Hence Tanya's number is 70.

Now that you've seen how bipartite graphs can help you solve logic puzzles, here's a final challenge: use bipartite graphs to design a new logic puzzle of your own!

References

[1] Kenneth Kong's Facebook page, photo posted on 10 April 2015, https://www.facebook.com/kennethjianwenz/photos/a.173663129479243.1073741827.167504136761809/385751114937109/?type=1 (accessed 24 April 2015).

[2] "A Math Problem from Singapore Goes Viral: When is Cheryl's Birthday?", *New York Times*, 14 April 2015, http://www.nytimes.com/2015/04/15/science/a-math-problem-from-singapore-goes-viral-when-is-cheryls-birthday.html (accessed 24 April 2015).

[3] "Cheryl's Birthday: Singapore's Maths Puzzle Baffles World", BBC News, 14 April 2015, http://www.bbc.com/news/world-asia-32297367 (accessed 24 April 2015).

[4] "The maths question for Singapore teenagers that has stumped the world", *Sydney Morning Herald*, 14 April 2015, http://www.smh.com.au/world/the-maths-question-for-singapore-teenagers-that-has-stumped-the-world-20150414-1mki4q.html (accessed 24 April 2015).

[5] "My number", Tanya Khovanova, posted on her blog on 16 April 2015, http://blog.tanyakhovanova.com/2015/04/my-number/ (accessed 24 April 2015).