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Combinatorial Derivations of Familiar Identities

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Finding two ways to enumerate the same collection of objects can often give rise to useful formulae. For instance, the sum

$$1 + 2 + \cdots + n$$

can be interpreted as the maximum number of different handshakes between $n + 1$ people. The first person may shake hands with n other people. The next person may shake hands with $n - 1$ other people, not counting the first person again. Continuing like this gives the above sum. Another approach is to simply realise that each of the $n+1$ guests shakes hands with n other guests. However, this counts handshakes twice. Therefore,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

This article concerns a combinatorial argument that gives rise to the familiar formula for the sum of the first n squares,

$$1^2 + 2^2 + \cdots + n^2.$$

We begin our investigation by fixing $n \in \mathbb{N}$. Now consider the set of consecutive integers

$$S = \{1, 2, \ldots, n+1, n+2\}.$$

Suppose that $a, b, c \in S$. We will call (a, b, c) a *rising sequence in* S if $a < b < c$.

Example. There are four rising sequences in $\{1, 2, 3, 4\}$, namely:

$$(1,2,3) \quad (1,2,4) \quad (1,3,4) \quad (2,3,4)$$

Our aim is to count the number of rising sequences in S two different ways. This will lead to a surprising combinatorial identity.

Enumerating rising sequences

You might immediately see a simple solution to this problem. If so, good. But we're first going to take a more arduous approach. We fix the first two terms then consider the number of choices for the third term. For example, if the first two terms are 1 and 2, then there are n choices for the final term: any one of $\{3, \ldots, n+1, n+2\}$. Likewise, if the first two terms are 5 and 7, then there are $n - 7$ choices for the final term: any one of $\{8, \ldots, n+1, n+2\}$. This consideration leads to the table shown below. The entry in position (i, j) gives the number of rising sequences whose first two terms are i and j respectively.

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(i,j)	2	3	4	$\cdots$	n	$n+1$
1	n	$n-1$	$n-2$		2	1
2		$n-1$	$n-2$		2	1
3			$n-2$		2	1
4					2	1
$\vdots$					$\vdots$	$\vdots$
$n-1$					2	1
n						1

Summing the entries in the above table gives the total number of rising sequences as

$$1 \cdot n + 2 \cdot (n-1) + \cdots + (n-1) \cdot 2 + n \cdot 1.$$

A simpler approach

You might try find a nice algebraic derivation of a formula for the above sum. However, perhaps the following argument is more elegant. Every three element subset of S corresponds to a rising sequence. This is because any choice of three members from S can be arranged from least to greatest. Therefore, the number of rising sequences is $\binom{n+2}{3}$. It follows that

$$1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \cdots + (n-1) \cdot 2 + n \cdot 1 = \binom{n+2}{3}.$$

Incidentally, this simplifies to give the pleasing formula,

$$1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \cdots + (n-1) \cdot 2 + n \cdot 1 = \frac{n(n+1)(n+2)}{6}.$$

Sums of squares

We now use the formula established above to obtain the familiar formula for the sum of the first n squares,

$$1^2 + 2^2 + \cdots + n^2.$$

Indeed, we have,

$$\begin{aligned}
\frac{n(n+1)(n+2)}{6} &= 1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \cdots + (n-1) \cdot 2 + n \cdot 1 \\
&= \sum_{j=1}^{n} j(n-j+1) \\
&= \sum_{j=1}^{n} (jn - j^2 + j) \\
&= \sum_{j=1}^{n} (j(n+1) - j^2)
\end{aligned}$$

$$=(n+1)\sum_{j=1}^{n} j - \sum_{j=1}^{n} j^2$$
$$=\frac{n(n+1)^2}{2} - \sum_{j=1}^{n} j^2.$$

It follows that

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)^2}{2} - \frac{n(n+1)(n+2)}{6}$$
$$= \frac{n(n+1)(2n+1)}{6}.$$

We skipped a little of the boring algebra but certainly we've obtained the familiar formula.

Concluding remarks

So at heart, this derivation of the formula for the sum of squares is combinatorial in nature. We've enumerated the same collection of objects two different ways and then equated the results. Though perhaps there's a nicer combinatorial argument to be found. We leave this as a challenge to the reader!