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Parametric Representations of Polynomial Curves Using Linkages

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Suppose that you want to draw a large perfect circle on a piece of fabric. A simple technique might be to use a length of string and a pen as indicated in Figure 1: tie a knot at one end (A) of the string, push a pin through the knot to make the center of the circle, tie a pen at the other end (B) of the string, and rotate around the pin while holding the string tight.

Figure 1: Drawing a perfect circle

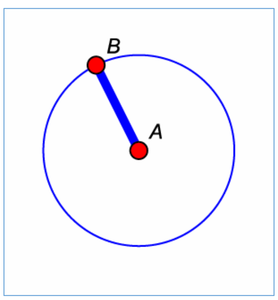
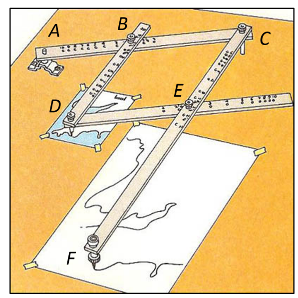


Figure 2: Pantograph



On the other hand, you may think about duplicating or enlarging a map. You might use a *pantograph*, which is a mechanical linkage connecting rods based on parallelograms. Figure 2 shows a draftsman's pantograph[3] reproducing a map outline at 2.5 times the size of the original.

As we may observe from the above examples, a combination of two or more points and rods creates a mechanism to transmit motion. In general, a *linkage* is a system of interconnected rods for transmitting or regulating the motion of a mechanism. Linkages are present in every corner of life, such as the windshield wiper linkage of a car[4], the pop-up plug of a bathroom sink[5], operating mechanism for elevator doors[6], and many mechanical devices. See Figure 3 for illustrations.

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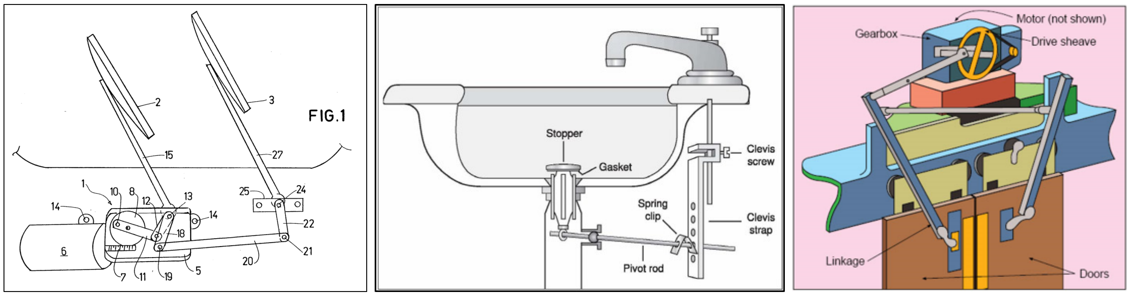
[3]http://www.daviddarling.info/encyclopedia/P/pantograph.html

[4]https://data.epo.org/publication-server/rest/v1.0/publication-dates/19951122/patents/EP0683076NWA1/document.html

[5]http://bit.ly/21V5FGw

[6]http://machinedesign.com/markets/motion-control-simulation-better-and-faster

Figure 3: Linkages: Windshield wiper, bathroom sink pop-up, elevator doors



It is useful to introduce the following definitions in analyzing systems of linkages.

Fixed point:	A point whose position remains fixed during linkage motion. It may act as a pivot.
Pivot:	A point on one or more rods around which the rods may rotate.
Rod:	A straight line segment connecting two distinct points. It can be moved and rotated but neither bent nor stretched.
Mover:	A particular point on a rod which rotates around a fixed point.
Driver:	The rod on which the mover lies.
Marker:	A point on a rod whose movement draws the desired curve.

For example, Figure 1 has the following characterization: A is a fixed point, the string or line segment AB is a rod, and B is both a mover and a marker.

Figure 2 consists of 4 rods and pivots: A is a fixed point; AC, BD, DE, CF are rods with fixed lengths; B, C, D, E are pivots; D is a mover and F is a marker.

Linkages all have in common that as the mover moves, the driver will cause the linkage to move, thus causing the marker to draw the given curve. This curve can therefore be seen as a function of the mover. In particular, we can express the position of the marker in terms of the angle of rotation of the driver, for instance in parametric form. For this purpose, we confine ourselves to linkages as mathematical drawing devices consisting of rods pivoted together so as to turn about one another, usually in the same plane or in parallel planes.

A circle in Figure 1 can be characterized as the set of all points with a fixed distance from a fixed point. To generalise, a circle could also be seen as the set of all points with fixed sum of distances from two *identical* points; if those two points are now allowed to be *distinct*, then we have an ellipse. If we consider a fixed difference of distances from two distinct points, then a hyperbola will be drawn. Then what kind of curve will be emerge if the product of distances is constant? What if the distances from two fixed points are equal? What if the distances from a fixed point and a fixed line are equal?

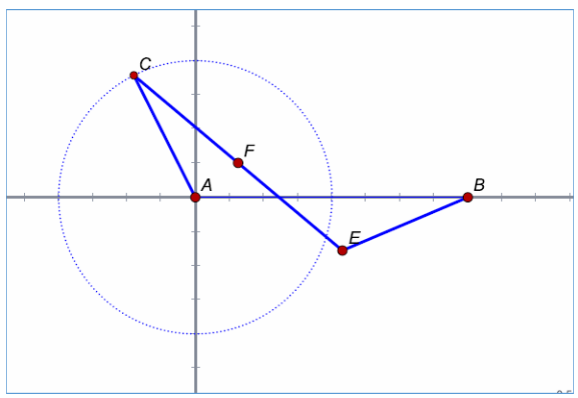
In this paper, we answer these three questions by constructing linkages which will draw the whole or part of the desired curves characterized by distances from point(s) and/or a line: the lemniscate of Bernoulli, the Peaucellier-Lipkin linkage for a straight line, and Yates' parabola. Basic algebra, geometry and trigonometry are used to find parametric representations of the loci (i.e., positions) of certain points in the system of linkages so that we can check that the linkages do in fact produce the claimed curves. The construction of linkages are carried out in *Cinderella* [2], an interactive geometry software which enables us to simulate physical objects. See [4] for a user manual.

1 Lemniscate of Bernoulli

You may not heard of the *lemniscate of Bernoulli* but you will easily recognize its curve as a figure-eight. Although not every figure-eight curve is a lemniscate, we can construct one in terms of distances from two fixed points.

Formally, the lemniscate of Bernoulli is defined to be set of loci of points whose product of distances from two fixed points at distance $2s$ from each other is s^2. A three-rod linkage as shown in Figure 4 produces a lemniscate of Bernoulli with the following characterization.

Figure 4: Lemniscate of Bernoulli linkage



Fixed points:	A, B
Pivots:	C, E
Mover:	C
Driver:	AC
Marker:	F is the midpoint of CE
Rods:	AC, BE, CE where $\|AC\| = \|BE\|$

To find a parametric representation of the marker, we assume that

$$A = (0,0)\,, \qquad B = (2s,0)\,, \qquad |AC| = |BE| = 1\,, \qquad |CE| = 2s$$

and that F is the midpoint of CE; see Figure 4. We measure the angle θ counterclockwise from the positive x-axis to the driver AC so that

$$C = (\cos\theta, \sin\theta)\,, \quad CB = (2s - \cos\theta, -\sin\theta)\,, \quad AE = t \cdot CB \quad \text{for some constant } t\,.$$

The point $E = t(2s - \cos\theta, -\sin\theta)$ satisfies $|BE| = 1$ which gives a quadratic equation in t:

$$\{t(2s - \cos\theta) - 2s\}^2 + (t\sin\theta)^2 = 1\,.$$

Since $t = 1$ is a trivial solution, it is not surprising to factor the equation and get the solution set as

$$\left\{\frac{4s^2 - 1}{4s^2 - 4s\cos\theta + 1}, 1\right\}\,.$$

Observing that the quadrilateral $ACBE$ is an anti-parallelogram, we discard the solution $t = 1$ to represent E as

$$E = \left(\frac{(4s^2-1)(2s-\cos\theta)}{4s^2 - 4s\cos\theta + 1}, \frac{-(4s^2-1)\sin\theta}{4s^2 - 4s\cos\theta + 1}\right)$$

and the marker F as

$$F = \frac{C+E}{2} = \left(\frac{1}{2}\left[\cos\theta + \frac{(4s^2-1)(2s-\cos\theta)}{4s^2 - 4s\cos\theta + 1}\right], \frac{\sin\theta}{2}\left[\frac{-(4s^2-1)}{4s^2 - 4s\cos\theta + 1} + 1\right]\right)\,.$$

After some calculation, we can check that if $s = \frac{1}{\sqrt{2}}$, then the curve satisfies the condition for a lemniscate of Bernoulli. In fact, if $|AC| = |BE| = 1$, $|CE| = \sqrt{2} = |AB|$, then the marker F can be simplified as

$$F = \left(\frac{1}{2}\left[\cos\theta + \frac{\sqrt{2} - \cos\theta}{3 - 2\sqrt{2}\cos\theta}\right], \frac{\sin\theta}{2}\left[1 - \frac{1}{3 - 2\sqrt{2}\cos\theta}\right]\right)$$

which gives the identity

$$|AF| \cdot |BF| = \frac{1}{2} = s^2\,.$$

For a simple geometric proof, see [1].

Figure 5: Lemniscate of Bernoulli curve

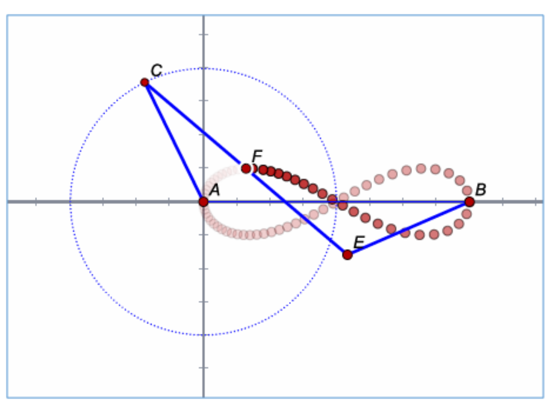
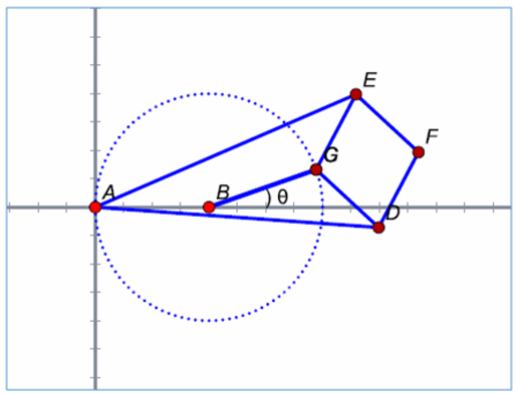


Figure 5 shows the lemniscate of Bernoulli drawn by marker F.

2 Peaucellier-Lipkin linkage

If we draw all points equidistant from two distinct fixed points, then we obtain a straight line which is a perpendicular bisector of the line segment of those two fixed points. A machine that will draw an exact straight line segment is the most basic and fundamental equipment in various fields of applications. In this section, we will focus on the *Peaucellier-Lipkin linkage* (Figure 6). It combines a line segment (BG) and a *Peaucellier inversor* consisting of 6 rods which will draw a vertical line segment.

Figure 6: Peaucellier-Lipkin linkage



Fixed points:	A, B
Pivots:	D, E, G, F
Mover:	G
Driver:	BG
Marker:	F
Rods:	AD, AE with $\|AD\| = \|AE\| = R$
	BG, EG with $\|BG\| = \|EG\|$
	EG, GD, DF, EF all of equal length r.

For a parametric representation, suppose that $A = (0, 0)$, $B = (1, 0)$ and $BG = 1$; see Figure 6. If θ is the angle between the positive x-axis and the driver BG measured counterclockwise, then $G = (\tilde{X}, \tilde{Y}) = (1+\cos\theta, \sin\theta)$. Two points D and E are solutions of the system of equations

$$(x - \tilde{X})^2 + (y - \tilde{Y})^2 = r^2\,, \qquad x^2 + y^2 = R^2\,.$$

Solving these equations, we obtain the equation of the line L passing through D and E:

$$2\tilde{X}x + 2\tilde{Y}y = R^2 - r^2 + \tilde{X}^2 + \tilde{Y}^2\,.$$

Observing that the line L is a perpendicular bisector of the line segment GF, we have a set of conditions for the marker $F = (X, Y)$:

$$\left(\frac{X + \tilde{X}}{2}, \frac{Y + \tilde{Y}}{2}\right) \text{ is on } L\,, \quad \frac{Y - \tilde{Y}}{X - \tilde{X}} \times (\text{slope of } L) = -1\,,$$

which can be simplified as

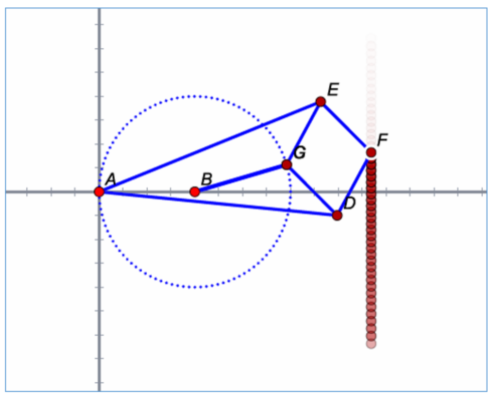
$$X\tilde{X} + Y\tilde{Y} = R^2 - r^2 \,, \qquad Y = \frac{\tilde{Y}}{\tilde{X}} \cdot X \,.$$

Now we solve for X and Y to get

$$X = \frac{1}{2}(R^2 - r^2) \,, \quad Y = \frac{\sin\theta}{2(1+\cos\theta)}(R^2 - r^2) \,.$$

This parametric representation confirms that the curve is really a part of a vertical line. The curve drawn by the marker is shown in Figure 7.

Figure 7: Locus of Peaucellier-Lipkin linkage



How can we determine the length of the vertical line segment in terms of R and r? What is the range of θ?

The maximum height h of the marker occurs when A, D, E, F and G are collinear. Using the Pythagorean Theorem, we get $h^2 = (R+r)^2 - (\frac{1}{2}(R^2 - r^2))^2$ which can be simplified as

$$h = (R+r)\sqrt{1 - \left(\frac{R-r}{2}\right)^2} \,.$$

In addition, the range of θ can be found from the relation below

$$\sin\theta = \sin(\pi - \theta) = (R-r) \cdot \frac{h}{R+r} = (R-r)\sqrt{1 - \left(\frac{R-r}{2}\right)^2}$$

which gives the range of θ as

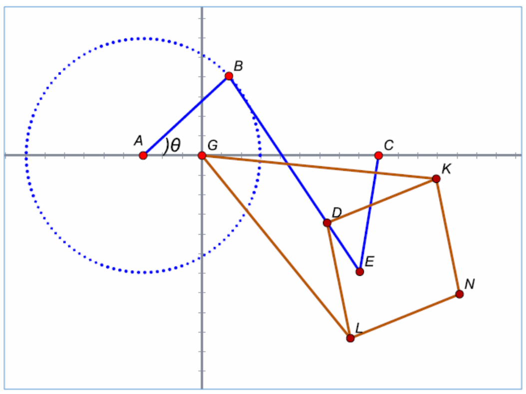
$$|\theta| \leq \sin^{-1}\left((R-r)\sqrt{1 - \left(\frac{R-r}{2}\right)^2}\right) \,.$$

Finally, for any specified line segment, we can locate two fixed points A, B on the perpendicular bisector of the line segment and determine the rod lengths R and r.

3 Yates' Parabola

A parabola has an geometrically intuitive interpretation: it is the set of all points that are at the same distance to a given line as to a fixed point. Van Schooten's rhombus linkage reveals this nature of the graph (see [5]). The construction, however, contains *sliders* in addition to rods and points. This is why we focus on another representation by Yates (see [3]) which combines a Peaucellier inversor and the lemniscate of Bernoulli as shown in Figure 8.

Figure 8: Yates' parabola linkage



Fixed points:	A, C, G
Pivots:	B, D, E, K, L, N
Mover:	B
Driver:	AB
Marker:	N
Rods:	AE, CE, DL, LN, NK, KD, each of length $2\lvert AG\rvert = \frac{1}{2}\lvert AC\rvert$ GL and GK with $\lvert GL\rvert = \lvert GK\rvert = 4\lvert AG\rvert$ BD and DE with $BD : DE = 3 : 1$

Suppose that $A = (-1, 0)$ and $C = (3, 0)$. We then have the following coordinates:

$$\begin{aligned}
B &= (-1 + 2\cos\theta, 2\sin\theta) \\
E &= \left(\frac{7 - 2\cos\theta}{5 - 4\cos\theta}, \frac{-6\sin\theta}{5 - 4\cos\theta}\right) \\
D &= (x, y) = \left(\frac{-2(1 + \cos\theta)(\cos\theta - 2)}{5 - 4\cos\theta}, \frac{-2(1 + \cos\theta)\sin\theta}{5 - 4\cos\theta}\right) \\
N &= (X, Y) = \left(\frac{12x}{x^2 + y^2}, \frac{12y}{x^2 + y^2}\right)
\end{aligned}$$

where θ is the angle between the driver AB and the positive x-axis measured counter-clockwise. The derivation of these points is similar to that of the lemniscate of Bernoulli and the Peaucellier-Lipkin linkages and is left for the reader.

We can easily check that $Y^2/(X-3)$ is the constant 4, so we get the following recognizable equation:

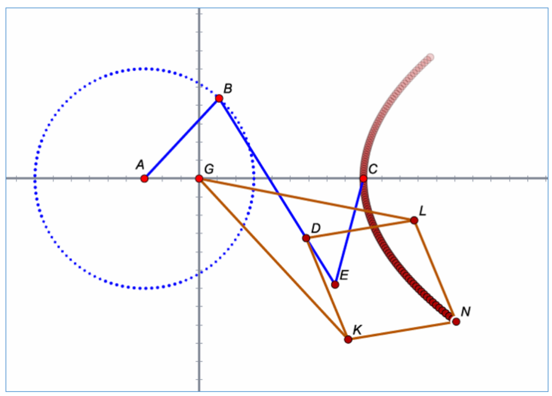
$$Y^2 = 4(X-3)\,.$$

The driver AB does not make a full revolution. In fact, observing that the extreme case occurs when $|AG| = 2$, we can determine the range of the angle θ as follows:

$$[-\cos^{-1}(-3+\sqrt{13}), \cos^{-1}(-3+\sqrt{13})]\,.$$

Thus our linkage draws part of a parabola (see Figure 9). The range of the parabola depends on the lengths and positions of the rods and points, respectively, of the linkage.

Figure 9: Yates' parabola



4 Conclusions

In this paper, we have investigated three specific linkages: the lemniscate of Bernoulli, the Peaucellier-Lipkin linkage for a straight line segment, and Yates' parabola. By treating the location of the marker as a function of a mover, we have constructed parametric representations and confirmed algebraically that each marker indeed draws the desired curve. The geometry software Cinderella is used to visualize the simulation.

The lemniscate of Bernoulli can be realized as a trace of a marker in a three-rod linkage, and the Peaucellier inversor enables us to create a straight line segment. Finally, a combination of a Peaucellier inversor and the lemniscate of Bernoulli produces a Yates' parabola which illustrates how linkages may be combined as building blocks to create new linkages.

Acknowledgments

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References

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[2] Cinderella, http://www.cinderella.de/tiki-index.php

[3] Irina Brinster and Jumana Jumbe, *Linkage Simulation in Cinderella*, http://www.utdallas.edu/~sxb027100/creu09/linkageSite.html.

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