Parabola Volume 52, Issue 1 (2016)

Problems 1491–1500

Parabola incorporating Function would like to thank Sin Keong Tong for contributing problem 1495.

Q1491 Find the 400th digit after the decimal point in the expansion of

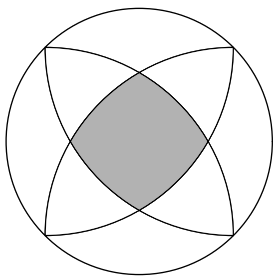
$$\left(\sqrt{20}+\sqrt{15}\right)^{2016} .$$

Q1492 Given any positive integer, we are allowed to double it, or to rearrange its digits. We can then perform either of these operations on the resulting number, and so on repeatedly. We start from the number 1. So we could obtain

$$1,\ 2,\ 4,\ 8,\ 16,\ 61,\ 122,\ 221,\ 442, \ldots$$

among other possibilities. Can we ever reach the number 2015? How about 2016?

Q1493 The following diagram (which may be familiar to Sydney readers) consists of one complete circle and four 90° arcs of other circles, whose centres are equally spaced around the circumference of the main circle. What is the area of the shaded region, as a fraction of the area of the whole circle?



Q1494 A certain country has very unusual laws regarding the construction of highways: between every pair of towns there must be a highway going in one direction but not in the other direction. Prove that there is a town which can be reached from every other town either directly, or through just one intermediate town.

Q1495 Consider the polynomial

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 ,$$

where the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are integers and an odd number of these coefficients, including the constant term a_0, are odd. Prove that $p(x)$ does not have an integer root.

Q1496 Find the minimum value of

$$\sqrt{1+x_1^2}+\sqrt{4+x_2^2}+\sqrt{9+x_3^2}\ ,$$

given that x_1, x_2, x_3 are real numbers with $x_1+x_2+x_3=8$.

Q1497 Let

$$p(x)=x^5-4x^3+4x^2+x+a\ ,\quad q(x)=x^4+3x^3+4x^2-x-15\ ,$$

where a is a real constant. Find all values of a for which $p(x)$ and $q(x)$ have a common root.

Q1498 We are to place 2015 balls in a circle; there are unlimited numbers of red and green balls available. If a ball is green, then exactly two of the next three balls (in a clockwise direction) must also be green; if the ball is red, then this is not true. Prove that our only option is to make all the balls red.

Q1499 Amanda and Belinda are playing a coin–tossing game in which one coin is thrown repeatedly until either TTH or HTH appears. In the former case Amanda wins, in the latter case Belinda wins. What is Amanda's winning probability?

Q1500 Obtain the number 1500 by using the operations $+$, $-$, $\times$, $\div$ on the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$, in that order. Brackets are allowed, but you cannot join digits to form multi–digit numbers: for example, you may not write 1 next to 2 and call this 12. One possibility is

$$(1+2+3+4)\times 5\times(6+7+8+9)\ .$$

See if you can find at least three more solutions.