Parabola Volume 52, Issue 2 (2016)

Pascal's many secrets

Peter G. Brown[1]

Pascal's Triangle arises in a very natural way when we expand the powers of $x + 1$. Stripping off the coefficients, we arrive at the well-known pattern shown below.

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}
$$

In this short article, I want to show you just a small sample of the huge number of remarkable patterns that can be found in this triangle of numbers.

The $\binom{n}{r}$ notation

You will have seen at school that the numbers in Pascal's Triangle can be written using the so-called binomial coefficients $\binom{n}{r}$, given by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}\,.$$

The variable n represents the row number (counting from 0) and r is the entry number across a given row, ranging from 0 to n.
So, for example, the number 56 in the bottom row shown above is the same as $\binom{8}{3}$.
Thus, the Binomial Theorem says that, for any positive integer n,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n\,. \qquad (*)$$

[1]Peter G. Brown is Senior Lecturer at the School of Mathematics and Statistics, UNSW Australia.

Fundamental identities

Two important identities immediately arise:

$$\binom{n}{r} = \binom{n}{n-r} \tag{A}$$

which simply says that the triangle is symmetric, and

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \tag{B}$$

which says that each entry in the triangle (with suitable conventions for the numbers at each end of a row) is the sum of the two numbers diagonally above.

Row sums

The most well-known pattern in the triangle is the row sum.

If we put $x = 1$ in (*), then we obtain

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n}$$

which tells us that the sum of the numbers in each row is a power of 2.

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&& \\
&&&&&&&1&&1&&&&&&& \\
&&&&&&1&&2&&1&&&&&& \\
&&&&&1&&3&&3&&1&&&&& \\
&&&&\boxed{1}&&\boxed{4}&&\boxed{6}&&\boxed{4}&&\boxed{1}&&&& \\
&&&1&&5&&10&&10&&5&&1&&& \\
&&1&&6&&15&&20&&15&&6&&1&& \\
&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}$$

$$1 + 4 + 6 + 4 + 1 = 16 = 2^4$$

Setting $x = -1$ gives

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^k\binom{n}{k} + \cdots + (-1)^n\binom{n}{n}$$

so the alternating sum across each row is zero.

This is not so surprising for odd numbered rows, since the triangle is symmetric, but it certainly tells us something for the even-numbered rows.

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & \boxed{1} & & \boxed{4} & & \boxed{6} & & \boxed{4} & & \boxed{1} & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}$$

$$1 - 4 + 6 - 4 + 1 = 0$$

A link to the Fibonacci numbers

The (shifted) Fibonacci numbers are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

and satisfy the recurrence

$$a_{n+2} = a_{n+1} + a_n \qquad \text{with} \qquad a_0 = a_1 = 0\,.$$

That is, each number is the sum of the previous two numbers.
We can make use of identity (B) above to derive an interesting result as follows.
Let $F(n)$ denote the sum

$$F(n) = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots$$

where we keep summing until the lower number exceeds the top one. The sum starts at one place on the far left of the triangle and moves across and up the triangle to the other side. Replacing n with $n+1$, we have

$$F(n+1) = \binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \binom{n-2}{3} + \cdots$$

and below, we again write

$$F(n) = \qquad\qquad \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots\,.$$

Adding these two equations and using (B), and recalling that $\binom{n+1}{0} = \binom{n+2}{0}$, we have

$$F(n+1) + F(n) = \binom{n+2}{0} + \binom{n+1}{1} + \binom{n}{2} + \cdots = F(n+2)\,.$$

This, combined with $F(0) = F(1) = 1$, shows us that this sum generates the *Fibonacci numbers* since both sequences have the same first two terms and satisfy the same second order recurrence.

This result can be demonstrated in terms of Pascal's Triangle.

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad \boxed{1} \\
1 \quad 5 \quad 10 \quad \boxed{10} \quad 5 \quad 1 \\
1 \quad 6 \quad \boxed{15} \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad \boxed{7} \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
\boxed{1} \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}$$

$1 + 7 + 15 + 10 + 1 = 34$, which is the 9th Fibonacci number.

Two way counting

In how many ways can 3 chapters be chosen from a book with 16 chapters?

There are clearly $\binom{16}{3}$ ways of choosing but we could also think of it as follows: If Chapter 1 is the first chapter to be chosen, then there are $\binom{15}{2}$ ways to choose the rest. If Chapter 2 is the first chapter to be chosen, then there are $\binom{14}{2}$ ways to choose the rest (since we have excluded Chapter 1 from the total list), and so on. Hence

$$\binom{16}{3} = \binom{15}{2} + \binom{14}{2} + \cdots + \binom{2}{2}.$$

If we now repeat the argument to find the number of ways to choose m chapters from n ($m < n$) chapters, then the following combinatorial identity is obtained.

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-2}{m-1} + \binom{n-3}{m-1} + \cdots + \binom{m-1}{m-1}.$$

This can be illustrated as follows:

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad \boxed{1} \\
1 \quad 3 \quad \boxed{3} \quad 1 \\
1 \quad 4 \quad \boxed{6} \quad 4 \quad 1 \\
1 \quad 5 \quad \boxed{10} \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad \boxed{15} \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad \boxed{21} \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad \boxed{56} \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}$$

$$21 + 15 + 10 + 6 + 4 + 1 = 56$$

Skip counting and complex numbers

Suppose that $n > 0$ and that we expand each of the expressions

$$(1+1)^n\,,\ (1+i)^n\,,\ (1-1)^n\,,\ (1-i)^n$$

using the Binomial Theorem. This gives

$$\begin{aligned}
2^n &= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}\\
(1+i)^n &= \binom{n}{0} + i\binom{n}{1} - \binom{n}{2} - i\binom{n}{3} + \cdots + i^n\binom{n}{n}\\
0 &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n}\\
(1-i)^n &= \binom{n}{0} - i\binom{n}{1} - \binom{n}{2} + i\binom{n}{3} + \cdots + (-i)^n\binom{n}{n}
\end{aligned}$$

Now add these equations and divide by 4 to obtain

$$\binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \cdots = \frac{1}{4}\Big(2^n + (1-i)^n + (1+i)^n\Big)\,.$$

It is easy to show, using the polar form, that $(1-i)^n + (1+i)^n = 2(\sqrt{2})^n \cos\left(\frac{n\pi}{4}\right)$.
Thus, we have

$$\binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \cdots = \frac{1}{4}\Big(2^n + 2(\sqrt{2})^n \cos\Big(\frac{n\pi}{4}\Big)\Big)\,.$$

Note that the right-hand side is, contrary to appearance, a positive integer.
This is a rather amazing and beautiful result.

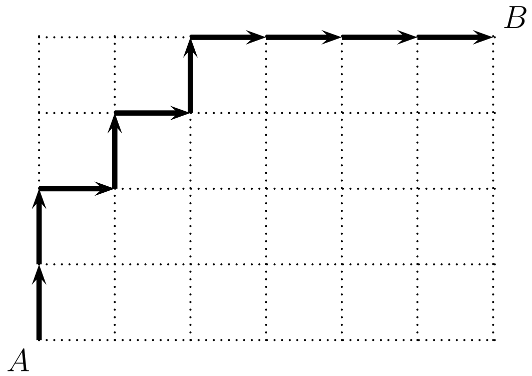
Furthermore, if n is one of the integers $2, 6, 10, 14, \ldots$ (increasing by 4), then $\cos\left(\frac{n\pi}{4}\right)$ equals 0, so the sum is 2^{n-2} in this case.

$$\begin{array}{c}
1\\
1 \quad 1\\
1 \quad 2 \quad 1\\
1 \quad 3 \quad 3 \quad 1\\
1 \quad 4 \quad 6 \quad 4 \quad 1\\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1\\
\boxed{1} \quad 6 \quad 15 \quad 20 \quad \boxed{15} \quad 6 \quad 1\\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1\\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1\\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1\\
\boxed{1} \quad 10 \quad 45 \quad 120 \quad \boxed{210} \quad 252 \quad 210 \quad 120 \quad \boxed{45} \quad 10 \quad 1
\end{array}$$

The sum of boxed numbers in the sixth row is $1 + 15 = 16 = 2^{6-2}$, as expected.
Similarly, the sum of boxed numbers in the tenth row is $1 + 210 + 45 = 256 = 2^{10-2}$.

Grids

On the grid shown, we wish to move from A to B by moving either to the right or upwards. How many paths are there from A to B?



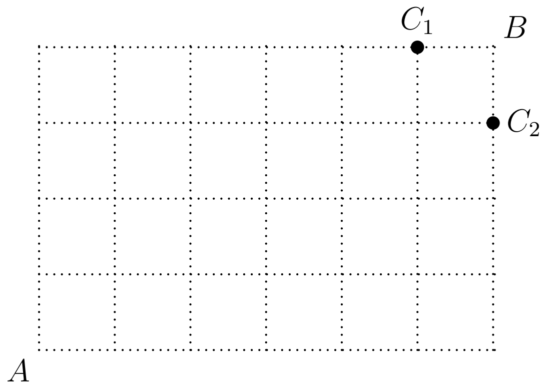
The diagram illustrates one possible path $UURURURRRR$.
Observe that there are 10 moves from A to B: 4 up and 6 right. To specify a particular path, we can say which of the twelve moves are up. Once we have decided which ones are up, the rest must be to the right. There are therefore $\binom{10}{4}$ possible paths.

Interestingly, we can now give a geometric proof of our identities (A) and (B). Suppose that we have an $k \times (n-k)$ grid, where $n \geq k$, and we wish to count the number of ways to get from A to B. There are n moves in total and we have to choose k of these moves to be towards the right, or alternatively $n-k$ moves in the upwards direction. Since these numbers must be the same, we have

$$\binom{n}{k} = \binom{n}{n-k}.$$

We can also prove that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

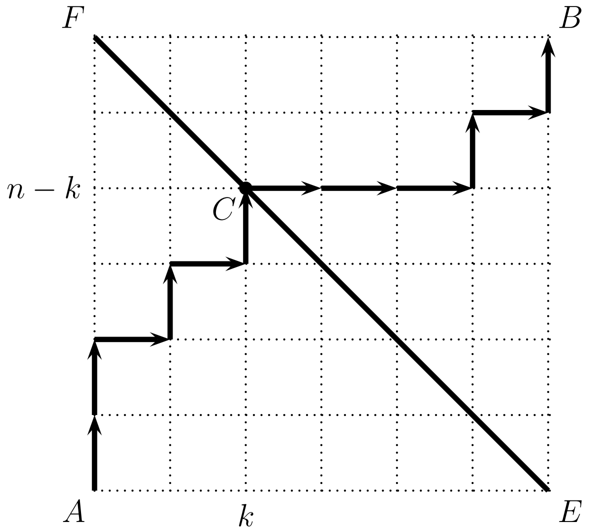


Each path from A to B must pass through exactly one of the points C_1 and C_2. Once we reach either of these points, there is only one way to finish. There are $\binom{n}{k}$ ways to get from A to B, while there are $\binom{n-1}{k-1}$ ways to reach C_1 and $\binom{n-1}{k}$ ways to reach C_2. Hence

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Sums of squares

On a square $n \times n$ grid, every path from A to B must pass through exactly one of the points on the diagonal EF. There are $\binom{2n}{n}$ paths from A to B. Take a general point C with coordinates $(k, n-k)$ on the diagonal. There are $\binom{n}{k}$ paths from A to C, and by symmetry, there are also $\binom{n}{k}$ paths from C to B. Thus there are $\binom{n}{k}^2$ paths from A to B passing through C.



Hence, summing over k from 0 to n, we have

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \sum_{k=0}^{n} \binom{n}{k}^2.$$

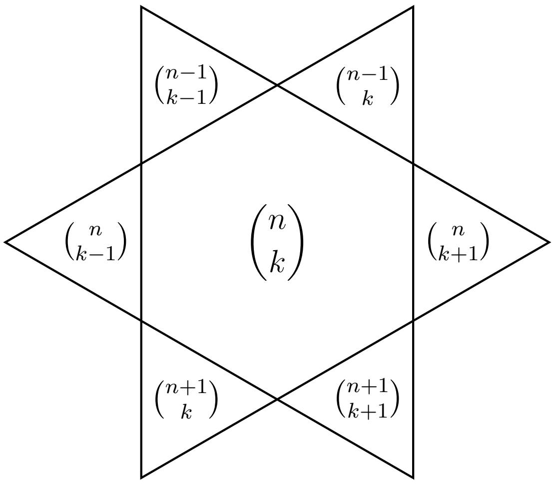
For instance for $n = 4$,

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & \boxed{1} & & \boxed{4} & & \boxed{6} & & \boxed{4} & & \boxed{1} & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & \boxed{70} & & 56 & & 28 & & 8 & & 1
\end{array}$$

$$1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70 = \binom{8}{4}.$$

The Star of David

Take any entry $\binom{n}{k}$ in the triangle, with say $n \geq 2$, $1 < k < n$, and draw a star around it.

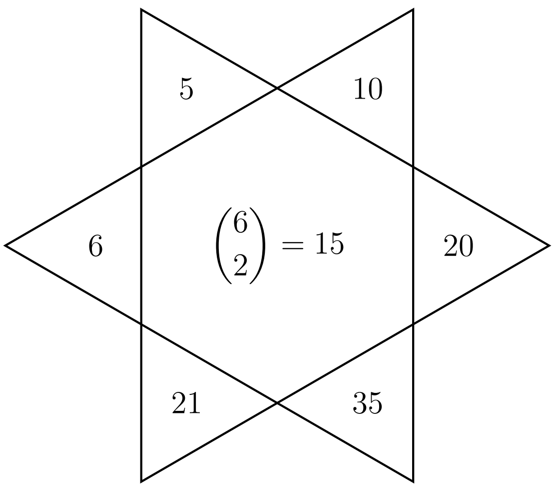


Then

$$\binom{n-1}{k-1}\binom{n+1}{k}\binom{n}{k+1} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}.$$

You can show this by dividing one side by the other and, using the formula for the binomial coefficients, then cancel the expression to arrive at 1. Have fun cancelling!

For example:



$$5 \times 20 \times 21 = 10 \times 6 \times 35$$

One for you

Sums of higher powers of binomial coefficients are connected (as are most sums involving binomial coefficients) to an amazing function known as the *hypergeometric function*. In particular, using that function, one can obtain:

$$\binom{n}{0}^3 - \binom{n}{1}^3 + \binom{n}{2}^3 - \cdots + (-1)^n \binom{n}{n}^3 = \begin{cases} 0 & \text{if } n = 2m-1 \\ \frac{(-1)^m (3m)!}{(m!)^3} & \text{if } n = 2m\,. \end{cases}$$

The case when n is odd is, of course, trivial, by the symmetry of the triangle.
Try proving the even case by induction!

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&& \\
&&&&&&&1&&1&&&&&&& \\
&&&&&&1&&2&&1&&&&&& \\
&&&&&1&&3&&3&&1&&&&& \\
&&&&\boxed{1}&&\boxed{4}&&\boxed{6}&&\boxed{4}&&\boxed{1}&&&& \\
&&&1&&5&&10&&10&&5&&1&&& \\
&&1&&6&&15&&20&&15&&6&&1&& \\
&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}$$

$$1^3 - 4^3 + 6^3 - 4^3 + 1^3 = 90 = (-1)^2 \frac{6!}{(2!)^3}$$

This is but a small sample of the many lovely patterns in Pascal's Triangle.