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# How to move a root of a cubic equation to the origin

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Suppose that a person wants to map a cubic equation in $x$ so that a given one of its roots (i.e. solutions) now lies in the origin ($x = 0$). Which mapping function is best suited for this task? Suppose that this person changes their mind and now wants to place the root at $x = 1$ for instance. Can the same mapping function be used, or should another one be used? We attempt to answer these questions in this paper.

There are many transformations, like linear transformations, Tschirnhaus transformations, Möbius transformations, etc., which can be employed as mapping functions. The linear transformation (expressed as $y = x + A$ where $A$ is some number — a constant), just transforms the cubic equation in $x$ to one in $y$ with no control over the placement of root of $y$. One may think of quadratic Tschirnhaus transformations (defined as $y = x^2 + Ax + B$ where $A$ and $B$ are constants) as mapping functions. These transformations are named after Ehrenfried Walther von Tschirnhaus who introduced them in 1683. However this type of transformation does not yield one-to-one mappings of polynomials, so it is also not suitable for our task of moving the root of cubic polynomials to the desired locations [1, 2].

We then look for a transformation which yields a one-to-one mapping of equations and their roots. In this context, we now examine whether the Möbius transformation, proposed by August Ferdinand Möbius in the nineteenth century, can be used to move the root of a cubic polynomial to the desired location. The general form of this transformation is

$$y = \frac{Ax + B}{Cx + D}$$

where $A$, $B$, $C$, and $D$ are constants such that $AD \neq BC$ [3].

## Moving the root to the origin

Consider the following cubic equation in $x$:

$$x^3 + ax + b = 0\,. \tag{1}$$

To move it, we plan to use the Möbius transformation of the form

$$y = \frac{x + c}{x + d} \tag{2}$$

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where $y$ is a new variable and $c$ and $d$ are as yet unknown numbers. The inverse of transformation (2) is

$$x = -\frac{dy - c}{y - 1}\,, \tag{3}$$

and this is used to substitute $x$ from (1) as shown below.

$$-\left(\frac{dy - c}{y - 1}\right)^3 - a\left(\frac{dy - c}{y - 1}\right) + b = 0\,. \tag{4}$$

Expanding and rearranging (4) in descending powers of $y$ and normalizing the coefficient of $y^3$ yields

$$y^3 + fy^2 + gy + h = 0 \tag{5}$$

where $f$, $g$, and $h$ are given by

$$\begin{aligned} f &= -\frac{3cd^2 + a(2d + c) - 3b}{d^3 + ad - b} \\ g &= \phantom{-}\frac{3c^2d + a(2c + d) - 3b}{d^3 + ad - b} \\ h &= -\frac{c^3 + ac - b}{d^3 + ad - b}\,. \end{aligned} \tag{6}$$

Notice that for finding the roots of the cubic equation (5) in a traditional manner, one has to force $f = 0$ and $g = 0$ to make (5) a binomial cubic equation in $y$, which can be readily solved. Also the conditions $f = 0$ and $g = 0$ facilitate determination of the two unknowns $c$ and $d$.

However, our aim is not to solve the cubic equation (5) *per se*, but to move a root of the equation to any desired location, which requires that one unknown be preserved for moving the root. Hence, we do not equate $f$ and $g$ to zero; rather, cubic equation (5) is rewritten as

$$\left(y + \frac{f}{3}\right)^3 + \left(g - \frac{f^2}{3}\right)y + h - \frac{f^3}{27} = 0\,, \tag{7}$$

and we apply the substitution

$$z = y + \frac{f}{3} \tag{8}$$

to expression (7). This results in

$$z^3 + \left(g - \frac{f^2}{3}\right)z + h - \frac{f}{3}g + \frac{2f^3}{27} = 0\,. \tag{9}$$

Notice that equating the coefficient of $z$ in (9) to zero makes it the perfect cube

$$z^3 = k^3\,, \tag{10}$$

where

$$k^3 = \frac{f}{3}g - h - \frac{2f^3}{27}\,, \tag{11}$$

This also yields a quadratic equation in $d$ after expanding and rearranging the expression $g - (f^2/3) = 0$ using (6):

$$3ad^2 - 9bd - a^2 = 0\,. \tag{12}$$

Solving (12), we find that

$$d = \frac{3b}{2a} \pm \sqrt{\frac{4a^3 + 27b^2}{12a^2}}\,. \tag{13}$$

Now, the expression for $k^3$ in (11) has to be expanded using the expressions for $f$, $g$, and $h$ given in (6). However before doing that, we rewrite the expressions in (6) as below, to ease the further algebraic manipulations.

$$\begin{aligned} f &= -3(cd^2 + [a(c+2d)/3] - b\}/D \\ g &= \quad 3(c^2d + [a(2c+d)/3] - b\}/D \\ h &= -(c^3 + ac - b)/D\,, \end{aligned} \tag{14}$$

where $D = d^3 + ad - b$. We apply the expressions given in (14) to (11) and obtain the following expression for $k^3$:

$$\begin{aligned} k^3 = \frac{1}{D^3}\Bigg(&2\left(cd^2 + \frac{a}{3}(c+2d) - b\right)^3 + D^2(c^3 + ac - b) \\ &- 3D\left(cd^2 + \frac{a}{3}(c+2d) - b\right)\left(c^2d + \frac{a}{3}(2c+d) - b\right)\Bigg) \end{aligned} \tag{15}$$

A close observation of (15) reveals that it is a cubic polynomial in $c$; thus (15) can be expressed as:

$$k^3 = (mc^3 + nc^2 + pc + q)/D^3, \tag{16}$$

where $m$, $n$, $p$, and $q$ are the coefficients of the cubic polynomial in (16). Since deriving expressions for these coefficients by expanding and arranging (15) at one stretch is cumbersome, we do it in a phased manner. First, from (15) we collect all $c^3$ terms, resulting in an expression for $m$:

$$m = bd^3 + (2a^2/3)d^2 - abd + (2a^3/27) + b^2\,. \tag{17}$$

Similarly, collection of all $c^2$ terms yields an expression for $n$; collection of all $c$ terms yields an expression for $p$; and collection of all constant terms yields an expression for $q$, as shown below.

$$\begin{aligned} n &= -3d[bd^3 + (2a^2/3)d^2 - abd + (2a^3/27) + b^2] \\ p &= \quad 3d^2[bd^3 + (2a^2/3)d^2 - abd + (2a^3/27) + b^2] \\ q &= -d^3[bd^3 + (2a^2/3)d^2 - abd + (2a^3/27) + b^2]\,. \end{aligned} \tag{18}$$

To our pleasant surprise, $m$ emerges as common factor in $n$, $p$, and $q$. Denoting $m = E^3$ and using it in (16) yields

$$k^3 = (E/D)^3(c^3 - 3dc^2 + 3d^2c - d^3)$$

which can be written in compact form as

$$k^3 = \big((E/D)(c-d)\big)^3 . \tag{19}$$

Use of (19) to eliminate $k^3$ from (10) results in $z^3 = \big((E/D)(c-d)\big)^3$, whose principal cube root is obtained as

$$z = (E/D)(c-d) . \tag{20}$$

Using (8) we eliminate $z$ from (20) and obtain a root of $y$ as

$$y = (E/D)(c-d) - f/3 . \tag{21}$$

Using (14) we eliminate $f$ from (21) and simplify, which yields

$$y = Fc + G \tag{22}$$

where $F$ and $G$ are given by

$$F = (3E + 3d^2 + a)/(3D) \qquad \text{and} \qquad G = (2ad - 3b - 3dE)/(3D) . \tag{23}$$

Notice from (22) that the unknown $c$ is preserved for placing a root of $y$ at any desired location. For example, to move the root of $y$ to $y = 0$, choose $c = -G/F$, and the corresponding root of $x$ is determined from (3) as $x = -c$.

Notice an interesting situation when we attempt to move a root to $y = 1$. From (22) and (23) we obtain $c = d$. The same result is obtained using $y = 1$ in (2), which renders the expression (3) to be indeterminate. Hence, $c \neq d$ and so $y \neq 1$. Therefore, the root cannot be moved to $y = 1$ with this form of the transformation.

As a numerical example, consider the cubic equation $x^3 - 6x - 9 = 0$. To move a root to a desired location (except to $y = 1$) using Möbius transformation (2). We determine two values of $d$ from (12): 4 and 0.5. We can use either value of $d$; let $d = 4$. Next, using the expression, $D = d^3 + ad - b$, we determine $D = 49$, and from (17) we get $m = -343$ and thus $E = -7$. From the expressions in (23), $F$ and $G$ are determined as $0.142857143$ and $0.428571429$. For moving a root of $y$ to $y = 0$, choose $c = -G/F = -3$, and the corresponding root of $x$ is determined from (3) to be 3.

Let us now take $d = 0.5$ and determine $D$, $m$, and $E$: $6.125$, $42.875$, and $3.5$. From (23) we determine $F$ and $G$ as $0.285714286$ and $0.857142857$. With these values, $c$ is determined to be $-3$ for moving the root of $y$ to the origin, and the corresponding root of $x$ is obtained from (3) to be 3.

**Moving the root to $x = 1$**

It is now clear that the Möbius transformation of the form (2) cannot move the root to $y = 1$. Let us consider another form of Möbius transformation,

$$y = r/(x+s), \tag{24}$$

where $r$ and $s$ are the two unknowns. Our aim is to find out whether the transformation (24) can move the root of $y$ to the $y = 1$ position. Expressing (24) as

$$x = (r - sy)/y \tag{25}$$

and using it in (1) to substitute $x$, and proceeding in the same fashion as before, we obtain an expression for a root of $y$ as

$$y = Hr\,, \tag{26}$$

where $H$ is given by

$$H = \frac{\Big(\big(s^3+as-b\big)^2 - \big(s^2+(a/3)\big)^3\Big)^{1/3} + s^2 + (a/3)}{s^3+as-b}, \tag{27}$$

and so

$$s = \frac{3b}{2a} \pm \sqrt{\frac{4a^3+27b^2}{12a^2}}\,. \tag{28}$$

The derivations of expressions (26), (27), and (28) are left as exercises to the readers. Notice from (26) that choosing $r = 1/H$ places a root of $y$ at $y = 1$; however, with the transformation of the form (24), we cannot move the root to the origin ($y = 0$).

Let us consider the same numerical example of the cubic equation $x^3 - 6x - 9 = 0$ for moving the root to $y = 1$ using the transformation (24). From (28), we obtain two values of $s$, namely $4$ and $0.5$. One may use either value of $s$. Using $s = 4$, we determine $H = \frac{1}{7}$ from (27). From (26), we know that $r = 7$ for moving the root to $y = 1$. Using (25) we obtain the corresponding root $x = 3$. Moving the root with the other value of $s$ is left to the readers as an exercise.

**Plots of cubic polynomials**

Figure 1 shows the plot of cubic polynomial $P(x) = x^3 - 6x - 9$. The cubic curve intersects the $x$-axis at only one point ($x = 3$), indicating that $P(x)$ has only one real zero, while the other two zeros are the complex conjugates $x = \frac{1}{2}(-3 \pm \sqrt{3}\, i)$.

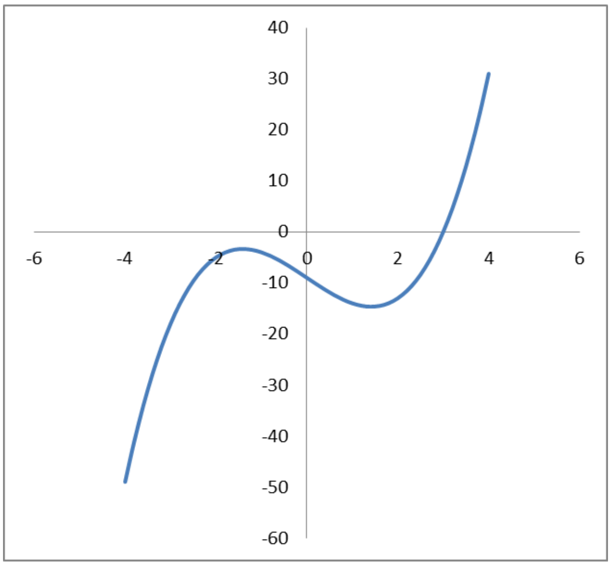


Figure 1: $P(x) = x^3 - 6x - 9$

To obtain the cubic equation in $y$ through the transformation (2), we apply the values $c = -3$ and $d = 4$ (see the numerical example in the previous section) to the expressions (6), and find that $f = 3$, $g = 3$, and $h = 0$. Now, the cubic equation in $y$ is obtained from (5) as follows: $y(y^2 + 3y + 3) = 0$. So, the cubic polynomial $Q(y) = y(y^2 + 3y + 3)$ has a zero placed at the origin $y = 0$.

Figure 2 shows the plot of the polynomial $Q(y)$. This polynomial intersects the $y$-axis at only one point, namely $y = 0$, implying that the cubic equation $Q(y) = 0$ has one real zero and the other two zeros are complex conjugates; these are $y = \frac{1}{2}(-3 \pm \sqrt{3}\, i)$.

Thus, we notice that the transformation (2) maps the root $x_1 = 3$ to $y_1 = 0$, the root $x_2 = \frac{1}{2}(-3 + \sqrt{3}\, i)$ is mapped to $y_2 = \frac{1}{2}(-3 + \sqrt{3}\, i)$; and the root $x_3 = \frac{1}{2}(-3 - \sqrt{3}\, i)$ is mapped as $y_3 = \frac{1}{2}(-3 - \sqrt{3}\, i)$.

Using $c = -3$ and $d = 0.5$ (another value of $d$; see the numerical example from previously) and (6), we determine that $f = -6$, $g = 12$, and $h = 0$. From (5), the cubic equation in $y$ is obtained as $y(y^2 - 6y + 12) = 0$. Thus transformation (2) maps the polynomial $P(x)$ as $R(y) = y(y^2 - 6y + 12)$, which has a zero at $y = 0$, and the other two zeros are $y = 3 \pm \sqrt{3}\, i$.

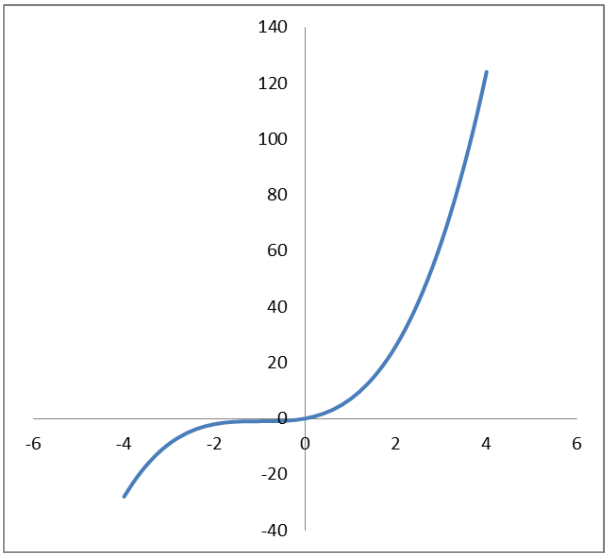


Figure 2: $Q(y) = y(y^2 + 3y + 3)$

Figure 3 shows the plot of the cubic polynomial $R(y)$. The roots $x_1$, $x_2$, and $x_3$ are mapped as $y_1 = 0$, $y_2 = 3 + \sqrt{3}\,i$ and $y_3 = 3 - \sqrt{3}\,i$ respectively.

The interested reader may plot the cubic polynomial in $y$ using the transformation (24) and see how the roots of $x$ are mapped in the $y$ domain.

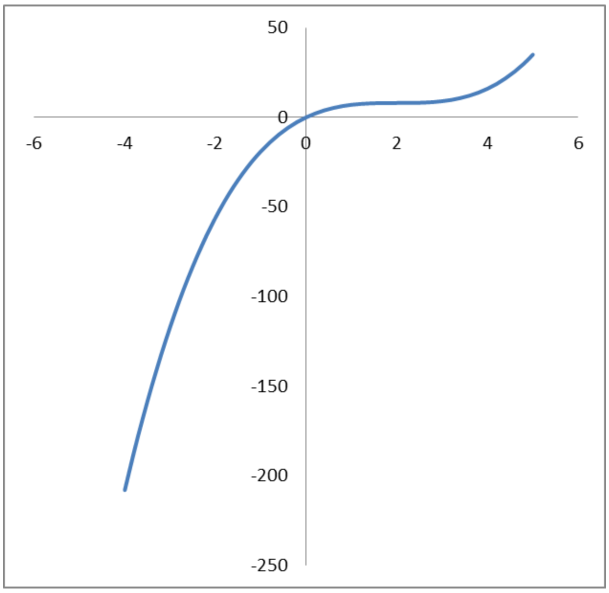


Figure 3: $R(y) = y(y^2 - 6y + 12)$

**Summary**

In this paper, we have shown that two different forms of Möbius transformation are required for moving a root of cubic to the origin $x = 0$ and to $x = 1$. In particular, the one which is used to place the root at the origin is $y = (x+c)/(x+d)$, where $c$ and $d$ are unknown numbers initially, and the unknown $c$ is preserved till the end to facilitate moving the root to the origin. The transformation used for placing the root at unity $x = 1$ has the form $y = r/(x + s)$, where $r$ and $s$ are unknowns to start with, and the unknown $r$ is preserved for placing the root at the unity.

The plots of cubic polynomials in $x$ and $y$ domains illustrate a root of $x$ being moved to the origin in the $y$ domain through the Möbius transformation.

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