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Divisibility of integers

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Generic rules for divisibility by small integers in the decimal system are well known and commonly used due to their simplicity. For instance, a number is divisible by 2 (i.e., it is even) if its last digit is divisible by 2; a number is divisible by $2^2 = 4$ if its last 2 digits are divisible by 4; and analogous rules are true for divisibility of 5 and $5^2 = 25$ as well. An integer is divisible by 3 if 3 divides the sum of its digits; this rule is also true for divisibility by 9. Similarly, an integer is divisible by 11 if 11 divides the alternating-signed sum of its digits.

In the text that follows, these rules are extended to integers n in an arbitrary base β. If

$$n = \sum_{i=0}^{N} a_i \beta^{N-i} \qquad \text{where } 0 \le a_i < \beta \quad \text{and } a_0 \neq 0\,,$$

then in base β, the integer n has digits $a_0, a_1, \ldots, a_N$, and we write $(n)_\beta = (a_0 a_1 \ldots a_N)_\beta$.

For instance, $(315)_{10}$ represents the number 315 in the usual base 10, whereas

$$315 = \mathbf{1} \times 6^3 + \mathbf{2} \times 6^2 + \mathbf{4} \times 6^1 + \mathbf{3} \times 6^0\,,$$

so in base 6, this number is represented as $(1243)_6$.
In general, $(n)_\beta$ is determined as follows:

$$\begin{aligned} N &= \left[\frac{\log n}{\log \beta}\right] \\ a_i &= \left[\frac{b_i}{\beta^{N-i}}\right], \quad i = 0, 1, \ldots, N, \quad b_0 = N, \\ b_{i+1} &= b_i - a_i \beta^{N-i}\,. \end{aligned}$$

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Proposition Consider the integer $(n)_\beta = (a_0 a_1 \dots a_N)_\beta$ in base β.

1. If $m|\beta$ and $m^j|(n)_{j,\beta}$ for some j with $1 \le j \le N$, where

$$(n)_{j,\beta} := (a_{N-j+1} \dots a_{N-1} a_N)_\beta = \sum_{i=1}^{j} a_{N-j+i} \beta^{j-i}\,.$$

   then $m^j|(n)_\beta$. In particular, $\beta^j|(n)_\beta$ only if $a_{N-j+i} = 0$ for $i = 1, 2, \dots, j$.

2. If $m|(\beta - \lambda)$ and $m|(n)_{\lambda'}$ where

$$(n)_{\lambda'} = \sum_{i=0}^{N} a_i (\lambda')^{N-i}\,, \qquad \text{where} \quad \lambda' = \lambda - m\left[\frac{\lambda - 1}{m}\right], \tag{1}$$

   then $m|(n)_\beta$.

   Hence, if $m|(\beta - 1)$ and $m \mid \sum_{i=0}^{N} a_i$, then $m|(n)_\beta$;

   if $m|(\beta + 1)$ and $m \mid \sum_{i=0}^{N} (-1)^i a_i$, then $m|(n)_\beta$;

   if β is odd and either $2 \mid \sum_{i=0}^{N} a_i$ or $2|\sum_{i=0}^{N} (-1)^i a_i$, then $2|(n)_\beta$.

Examples

1. $3^2|(1243)_6$ since $3^2|(43)_6$ but $3^3 \nmid (1243)_6$ since $3^3 \nmid (243)_6$
2. $5|(4(15)(11))_{16}$ since $5|(4 + 15 + 11)$
3. $3|(16107)_8$ since $3|(1 - 6 + 1 - 0 + 7)$
4. $2 \nmid (123456)_5$ since $2 \nmid (1 + 2 + 3 + 4 + 5 + 6)$
5. $7|75075$ since $(75075)_3 = 728$, $(728)_3 = 77$
   $13|75075$ since $(75075)_{-3} = 416$, $(416)_{-3} = 39$
6. $19|19703$ since $19703 = (2953)_{20}$ and $19|(2 + 9 + 5 + 3)$

or $19|19703$ since $19703 = (36(14)(11))_{18}$ and $3 - 6 + 14 - 11 = 0$.

Proof.

1. We have

$$\frac{(n)_\beta - (n)_{j,\beta}}{m^j} = \left(\frac{\beta}{m}\right)^j ((n)_\beta \backslash (n)_{j,\beta}) \ ,$$

   where

$$((n)_\beta \backslash (n)_{j,\beta}) = (a_0 a_1 \ldots a_{N-j})_\beta = \sum_{i=0}^{N-j} a_i \beta^{N-j-i} \ .$$

   In particular,

$$\frac{(n)_\beta - (n)_{j,\beta}}{\beta^j} = ((n)_\beta \backslash (n)_{j,\beta}) \ , \qquad (n)_{j,\beta} < \beta^j \ .$$

   Hence $\beta^j | (n)_\beta$ only if $(n)_{j,\beta} = 0$.

2. We have

$$(n)_\beta - (n)_{\lambda'} = \sum_{i=0}^{N-1} a_i \left(\beta^{N-i} - (\lambda')^{N-i}\right) \ .$$

   Hence,

$$\frac{(n)_\beta - (n)_{\lambda'}}{m} = \left(\frac{\beta - \lambda}{m} + \left[\frac{\lambda - 1}{m}\right]\right) \sum_{i=0}^{N-1} a_i \sum_{j=0}^{N-i-1} (\lambda')^{N-i-j-1} \beta^j \ .$$

This completes the proof. □