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Problems 1501–1510

Parabola incorporating Function would like to thank Sin Keong Tong for contributing problem 1501, and Farid Haggar for problem 1508.

Q1501 Find the sum of the sum of the sum of the digits for the number 2016^{2016}.

Q1502 Recall the country described in Problem 1494: between every pair of towns there is a highway going in one direction but not in the other direction. As in the solution (later this issue), a town is called "central" if it can be reached from every other town either directly, or with just one intermediate town. If the number of towns in the country is odd, then show how to arrange the highways in such a way that **every** town is central.

Q1503 Use the method of Problem 1496 (solution later this issue) to solve the following problems.

(a) Find the maximum value of

$$\sqrt{1-x_1^2}+\sqrt{4-x_2^2}+\sqrt{9-x_3^2}\,,$$

given that x_1, x_2, x_3 are positive real numbers with $x_1+x_2+x_3=1$.

(b) Find the minimum value of

$$\sqrt{(x-1)^2+(x^2-2)^2}+\sqrt{(x-3)^2+(x^2-4)^2}\,,$$

where x is a real number. Also, find the value of x which gives this minimum.

Q1504 A sequence of numbers $a_1, a_2, a_3, \ldots$ is defined by the properties

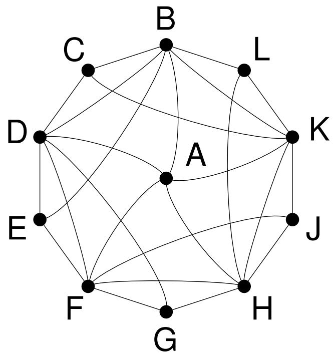
$$a_1=3\,,\quad a_{n+1}=a_n^2-2 \text{ for } n\geq 1\,.$$

Find the value which

$$\frac{a_n}{a_1a_2a_3\cdots a_{n-1}}$$

approaches as n becomes very large.

Q1505 It's easy to see that we can start at the point A in the following diagram, then travel along the lines in such a way as to visit every one of the labelled points without repeating any points: for example, $ABCDEFGHJKL$. But is it possible to do the same thing starting at B?



Q1506 The following question is probably not all that hard, but see if you can solve it in three different ways: (a) by plain algebra; (b) by drawing a graph; (c) by calculus.

Let a, b, c be real numbers such that $a < b < c$. Show that the equation

$$\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} = 0$$

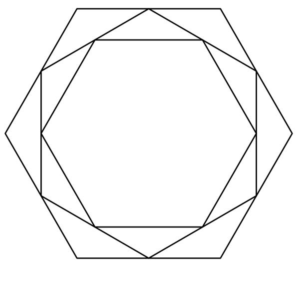
has one solution x between a and b, and another between b and c.

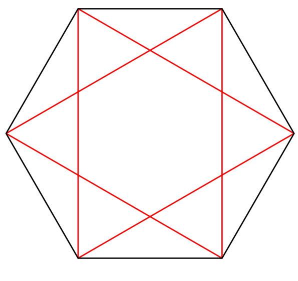
Q1507 Form a six-digit number from a standard keypad

1	2	3
4	5	6
7	8	9

by going along any row, column, or diagonal and then back again. For example, we could get 258852 or 951159, among others. Prove that every number which can be obtained in this way is a multiple of 1221.

Q1508 In the left-hand diagram we begin with a regular hexagon and join the midpoints of the sides to form a smaller regular hexagon; we then repeat the same procedure a second time to obtain a still smaller hexagon; we could repeat this step as many times as we wish.





In the right-hand diagram we begin with another hexagon, of the same size as the original hexagon on the left, and join every second vertex so that a smaller hexagon is formed inside the original. We only do this once. How many times must we perform the "midpoint" construction so that the resulting hexagon will lie inside the small hexagon in the right-hand diagram?

Q1509 If $0 < \theta < \frac{1}{2}\pi$, then show that

$$\log_{1+\sin\theta}(\cos\theta) + \log_{1-\sin\theta}(\cos\theta) = 2\log_{1+\sin\theta}(\cos\theta)\log_{1-\sin\theta}(\cos\theta)\,.$$

Q1510 If n and r are positive integers, then the set $S = \{1, 2, \ldots, n\}$ has $\binom{n}{r}$ subsets of size r. Here $\binom{n}{r}$ is the binomial coefficient, sometimes written as $C(n, r)$ or ${}^nC_r$.

(a) How many subsets of S have size r and largest element k?

(b) Prove that

$$\binom{r-1}{r-1} + \binom{r}{r-1} + \binom{r+1}{r-1} + \cdots + \binom{n-1}{r-1} = \binom{n}{r}\,.$$

(c) Show that

$$k\binom{m}{k} = m\binom{m-1}{k-1}\,.$$

(d) If from every size r subset of S we write down the largest element, then what is the total of the numbers we have written?
(*Hint*: Simplify your expression by using the results of (b) and (c).)