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Solutions 1491–1500

Q1491 Find the 400th digit after the decimal point in the expansion of

$$\left(\sqrt{20}+\sqrt{15}\right)^{2016}.$$

SOLUTION Consider

$$N=\left(\sqrt{20}+\sqrt{15}\right)^{2016}+\left(\sqrt{20}-\sqrt{15}\right)^{2016}.$$

Expanding by the Binomial Theorem,

$$N=\sum_{k=0}^{2016}\binom{2016}{k}\left(\sqrt{20}\right)^{k}\left(\sqrt{15}\right)^{2016-k}+\sum_{k=0}^{2016}(-1)^{k}\binom{2016}{k}\left(\sqrt{20}\right)^{k}\left(\sqrt{15}\right)^{2016-k}.$$

Now the terms with even values of k will be integers (since an even power of $\sqrt{a}$ is a whole power of a), while the terms with odd k in the first sum will cancel with the corresponding terms in the second sum. Therefore N is an integer, and we have

$$\left(\sqrt{20}+\sqrt{15}\right)^{2016}=N-\left(\sqrt{20}-\sqrt{15}\right)^{2016}.$$

Moreover,

$$\sqrt{20}-\sqrt{15}=\frac{20-15}{\sqrt{20}+\sqrt{15}}<\frac{5}{8};$$

a (fairly) easy calculation then shows that

$$\left(\sqrt{20}-\sqrt{15}\right)^{5}<\frac{3125}{32768}<\frac{1}{10}$$

and so

$$\left(\sqrt{20}-\sqrt{15}\right)^{2016}<\left(\sqrt{20}-\sqrt{15}\right)^{2015}<\frac{1}{10^{403}}.$$

So the decimal expansion of $\left(\sqrt{20}-\sqrt{15}\right)^{2016}$ begins with at least 403 zeros; when this is subtracted from the integer N, the part after the decimal point begins with at least 403 nines. So the 400th digit is a 9.

Q1492 Given any positive integer, we are allowed to double it, or to rearrange its digits. We can then perform either of these operations on the resulting number, and so on repeatedly. We start from the number 1. So we could obtain

$$1,\ 2,\ 4,\ 8,\ 16,\ 61,\ 122,\ 221,\ 442,\ldots$$

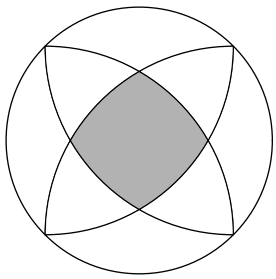
among other possibilities. Can we ever reach the number 2015? How about 2016?

SOLUTION Working backwards from 2015 with a bit of trial and error, we see that 2015 could be produced from 1052, which could come from 526, which could come from 256, and this is a power of 2 so it can be reached by doubling from 1. The full sequence is

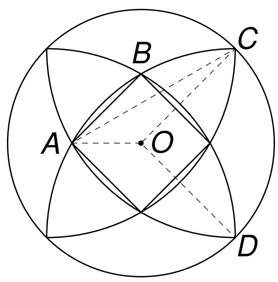
$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ 256,\ 526,\ 1052,\ 2015\,.$$

Note, however, that 2016 is a multiple of 3. It can only be obtained by doubling a number which is already a multiple of 3, or rearranging the digits of a number which is already a multiple of 3. Since the only number we are given initially is 1, which is not a multiple of 3, we can never reach any multiple of 3. Therefore it is not possible to get to 2016.

Q1493 The following diagram (which may be familiar to Sydney readers) consists of one complete circle and four 90° arcs of other circles, whose centres are equally spaced around the circumference of the main circle. What is the area of the shaded region, as a fraction of the area of the whole circle?



SOLUTION Label points as shown. We may as well assume that the radius of the main circle is 1; then the distance between any two adjacent



points of four equally spaced around its circumference is $\sqrt{2}$, and this is also the radius of the four 90° arcs. The shaded region consists of a square with side AB and four regions bounded by a circle of radius $\sqrt{2}$ and a chord of length AB. Each of the latter regions is a sector minus a triangle; the angle of the sector is 2θ, where $\theta = \angle ADO = \angle ACO$; so the area of each of these regions is

$$\frac{2\theta}{2\pi}\pi\big(\sqrt{2}\big)^2 - \frac{1}{2}AB(\sqrt{2}\cos\theta) = 2\theta - \frac{1}{\sqrt{2}}AB\cos\theta\,.$$

We need to find the length AB and the angle θ. First note that $\triangle ACD$ is equilateral, because its three sides are radii of equal-sized circles. Hence $\angle DAC = 60^\circ$ and $\angle OAC = 30^\circ$. Also, $\angle DOC$ is a right angle, so $\angle AOC = 135^\circ$. Hence

$$\theta = \angle ACO = 15^\circ = \frac{\pi}{12} \; ;$$

and since OC is a radius of the "main" circle, the sine rule in $\triangle AOC$ gives

$$AO = \frac{\sin 15^\circ}{\sin 30^\circ} = \frac{1}{2\cos 15^\circ} = \frac{1}{2\cos\theta} \, .$$

It is clear that $AB = \sqrt{2}\, AO$, and so the total shaded area is

$$\begin{aligned} AB^2 + 4\Bigl(2\theta - \frac{1}{\sqrt{2}} AB\cos\theta\Bigr) &= \frac{1}{2\cos^2\theta} + 8\theta - 2 \\ &= \frac{1}{1+\cos 2\theta} + \frac{2\pi}{3} - 2 \\ &= \frac{2\pi}{3} - \frac{2+2\sqrt{3}}{2+\sqrt{3}} \, , \end{aligned}$$

and this is a fraction

$$\frac{1}{\pi}\Bigl(\frac{2\pi}{3} - \frac{2+2\sqrt{3}}{2+\sqrt{3}}\Bigr) = 0.2006$$

of the main circle.

Q1494 A certain country has very unusual laws regarding the construction of highways: between every pair of towns there must be a highway going in one direction but not in the other direction. Prove that there is a town which can be reached from every other town either directly, or with just one intermediate town.

SOLUTION We'll refer to a town which can be reached from every other town either directly, or with just one intermediate town, as a "central" town. The claim that there exists a central town is clearly true if the country has only one town (as there are no other towns to cause any problems!), and if it has two towns (as one of the towns will have a highway from the other, and so the former is central). Assume that the claim is true for a town with n countries, and let x be the central town. Now add another town y, and construct highways between y and every other town. We shall prove that either x or y is a centre of the expanded country.

So, suppose that x is **not** a centre of the expanded country. Since we know that x is reachable in at most two steps from every "old" town, the only possible problem is with y. Therefore there is no highway $y \to x$, and no pair of highways $y \to \,?\, \to x$.

Now consider any town a other than y.

- If $a = x$, then there is no highway $y \to a$, so there is a highway $a \to y$.
- If there is a highway $a \to x$, then there is a pair $a \to x \to y$.

- If there is a pair $a \to b \to x$, then there is no highway $y \to b$, so there is a highway $b \to y$ and there is a pair $a \to b \to y$.

This shows that if a country with n towns has a central town, then so does a country with $n+1$ towns. By mathematical induction, no matter how many towns are in the country, then there must be a central town.

NOW TRY Problem 1502.

Q1495 Consider the polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 ,$$

where the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are integers and an odd number of these coefficients, including the constant term a_0, are odd. Prove that $p(x)$ does not have an integer root.

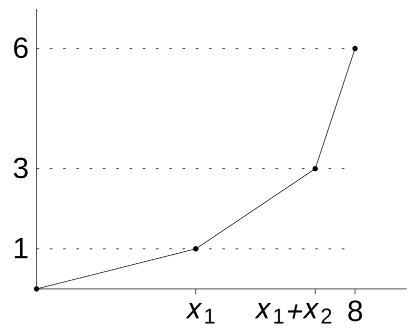
SOLUTION If x is even then every term $a_k x^k$ is even, except for a_0 which is odd. So the sum, which is $p(x)$, is odd. On the other hand, if x is odd, then $a_k x^k$ is odd if a_k is odd, even if a_k is even. So in this case $p(x)$ is the sum of some even numbers and an odd number of odd numbers, which is still odd. So when x is an integer $p(x)$ is always odd, therefore never zero, and $p(x)$ has no integer roots.

Q1496 Find the minimum value of

$$\sqrt{1 + x_1^2} + \sqrt{4 + x_2^2} + \sqrt{9 + x_3^2} \, ,$$

given that x_1, x_2, x_3 are real numbers with $x_1 + x_2 + x_3 = 8$.

SOLUTION The given expression is the distance from $(0, 0)$ to $(8, 6)$, going via an unspecified point $(x_1, 1)$ on the line $y = 1$ and then an unspecified point $(x_1 + x_2, 3)$ on the line $y = 3$, as shown in the diagram.



The minimum value is clearly the straight line distance from $(0, 0)$ to $(8, 6)$, which by Pythagoras' Theorem is 10.

NOW TRY Problem 1503.

Q1497 Let

$$p(x) = x^5 - 4x^3 + 4x^2 + x + a\ , \quad q(x) = x^4 + 3x^3 + 4x^2 - x - 15\ ,$$

where a is a real constant. Find all values of a for which $p(x)$ and $q(x)$ have a common root.

SOLUTION Dividing $p(x)$ by $q(x)$ to obtain a quotient and remainder, we have

$$p(x) = (x-3)q(x) + (x^3 + 17x^2 + 13x + a - 45)\ .$$

Now if $p(x)$ and $q(x)$ have a common root α, this is also a root of

$$\begin{aligned} r(x) &= p(x) - (x-3)q(x) \\ &= x^3 + 17x^2 + 13x + a - 45\ . \end{aligned}$$

Repeating the procedure, α is also a root of

$$\begin{aligned} s(x) &= q(x) - (x-14)r(x) \\ &= 229x^2 + (226-a)x + (14a - 645) \end{aligned}$$

and of

$$\begin{aligned} t(x) &= 229^2 r(x) - (229x + a + 3667)s(x) \\ &= (a^2 + 235a + 696)x - (14a^2 - 1748a - 5370)\ . \end{aligned}$$

Now consider the quadratic $a^2 + 235a + 696$. It has roots -3 and -232. If $a = -3$, then we also have $14a^2 - 1748a - 5370 = 0$, so $t(x) = 0$, which has α as a root (every number is a root of the zero polynomial). So -3 is a potential value of a, which we shall check later. If $a = -232$, we find that $t(x)$ is a non–zero constant, which has no roots, so this case is ruled out. If a has neither of these values then we continue the reduction procedure one more time, eleiminating all the xs to give a polynomial $u(x)$ which still has α as a root. However it turns out (check it if you like, but use computer assistance – the algebra is horrible) that u is a quadratic in a with no real roots, so this is impossible.

The only remaining possibility is $a = -3$, and this does in fact work, since then

$$p(x) = x^5 - 4x^3 + 4x^2 + x - 3 = (x^3 - x^2 + 1)(x^2 + x - 3)$$

and

$$q(x) = x^4 + 3x^3 + 4x^2 - x - 15 = (x^2 + 2x + 5)(x^2 + x - 3)$$

do have a common root (in fact, two common roots).

Q1498 We are to place 2015 balls in a circle; there are unlimited numbers of red and green balls available. If a ball is green, then exactly two of the next three balls (in a clockwise direction) must also be green; if the ball is red, then this must not be the case. Prove that our only option is to make all the balls red.

SOLUTION Suppose that there is at least one green ball. Then two of the next three balls are green, so we have three greens within four consecutive balls. There must therefore be two adjacent greens. The first of these must be followed by two greens and a red (including the green we have already), so we have

$$GGGR \quad \text{or} \quad GGRG\,.$$

In both cases the second ball must be followed by two greens and a red, so we have

$$GGGRG \quad \text{or} \quad GGRGG\,;$$

and now in the first case ball 3 must be followed by two greens and a red, while in the second case ball 3 must **not** be followed by two greens and a red, so we have

$$GGGRGG \quad \text{or} \quad GGRGGG\,.$$

But both options currently end with two adjacent greens, so the whole argument now repeats and we get

$$GGGRGGGR\cdots \quad \text{or} \quad GGRGGGRG\cdots\,.$$

Thus the whole set of balls consists of three greens and a red, repeated over and over; and since the number of balls is not a multiple of 4, this is impossible. Therefore there can be no green ball, and all balls are red.

Q1499 Amanda and Belinda are playing a coin-tossing game in which one coin is thrown repeatedly until either TTH or HTH appears. In the former case Amanda wins, in the latter case Belinda wins. What is Amanda's winning probability?

SOLUTION Let p_{HH} be Amanda's probability of winning if the previous two throws are HH, and define $p_{HT},\ p_{TH}$ and p_{TT} similarly.

- Suppose that the previous two throws are HH. Then there is a $\frac{1}{2}$ chance that the next throw is H, and after this Amanda's chance of winning is p_{HH}, and there is a $\frac{1}{2}$ chance that the next throw is T, and after this Amanda's chance of winning is p_{HT}. Therefore
$$p_{HH} = \tfrac{1}{2}p_{HH} + \tfrac{1}{2}p_{HT}\,.$$
- Suppose that the previous two throws are HT. Then there is a $\frac{1}{2}$ chance that the next throw is H, in which case Belinda has won and Amanda's winning chance is 0, and there is a $\frac{1}{2}$ chance that the next throw is T, after which Amanda's winning chance is p_{TT}. Therefore
$$p_{HT} = \tfrac{1}{2}p_{TT}\,.$$
- By similar arguments,
$$p_{TH} = \tfrac{1}{2}p_{HH} + \tfrac{1}{2}p_{HT} \quad \text{and} \quad p_{TT} = \tfrac{1}{2} + \tfrac{1}{2}p_{TT}\,.$$

These equations are easy to solve (starting with the last) and we get

$$p_{HH} = p_{HT} = p_{TH} = \tfrac{1}{2}\,, \quad p_{TT} = 1\,.$$

But the first two throws can be HH, HT, TH or TT, each with probability $\frac{1}{4}$, so Amanda's total winning probability is

$$\frac{1}{4}\Big(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+1\Big) = \frac{5}{8}\,.$$

Q1500 Obtain the number 1500 by using the operations $+$, $-$, $\times$, $\div$ on the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$, in that order. Brackets are allowed, but you cannot join digits to form multi–digit numbers: for example, you may not write "one" next to "two" and call it "twelve". One possibility is

$$(1+2+3+4)\times 5\times(6+7+8+9)\ :$$

see if you can find at least three more solutions.

SOLUTION Including the given expression, we have

$$\begin{aligned}1500 &= (1+2+3+4)\times 5\times(6+7+8+9)\\ &= (1+2+3+4)\times 5\times(-6\times 7+8\times 9)\\ &= (1\times(2+3)\times 4+5)\times 6\times(-7+8+9)\\ &= 1\div 2\div 3\times 4\times 5\times(6\times 7+8)\times 9\,.\end{aligned}$$