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Problems 1511–1520

Parabola thanks Professor Miklos N. Szilagyi for contributing Problem 1515.

Q1511 In a certain country (see Problems 1494 and 1502), between every pair of towns there is a highway going in one direction but not in the other direction. A town is called "central" if it can be reached from every other town either directly, or with just one intermediate town.

(a) Show that if there are 8 towns in this country, then it is possible for *every* town to be central.

(b) Show that the same is true for any number of towns except 2 or 4.

Q1512 Find all solutions of the simultaneous equations

$$x^2 + 4y^2 + z^2 = 20 \quad \text{and} \quad x + yz = 6\,.$$

Q1513 Divide the following array of numbers

2	2	1	2	1	1	1
2	4	3	1	1	2	2
1	1	2	1	1	2	1
5	1	3	1	6	1	1
1	3	3	1	1	1	4

into 11 connected regions, each containing numbers adding up to 6. A "connected region" means a set of squares in which every square is joined to some other square along an edge (not just a corner). No combination of numbers in a region may be used more than once. For example, you might use one of the following regions,

1	1	2	2

1	2
2	1

1	1
	2
	2

but you may not use more than one of them because they all contain the same numbers.

Q1514 Find two (non-constant) polynomials whose product is the polynomial

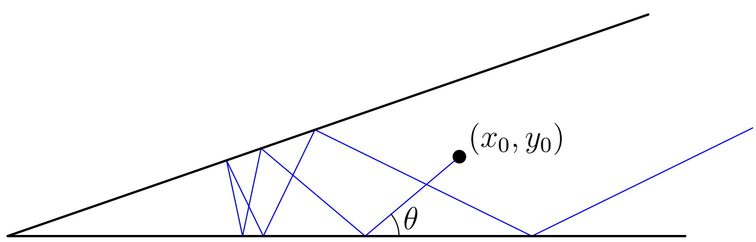
$$f(x) = 1 + x^{1010} + x^{1011} + x^{1012} + x^{1013} + \cdots + x^{2017}\,.$$

Q1515 Replace each letter by a different digit in the following long division in such a way that the working is correct. You may assume (as stated in the article by Prof Miklos N. Szilagyi earlier this issue) that no number begins with a zero.

```
         A C H G A
      -------------
H J | A B C D E F G
        H J
      -----
        C D D
        B F G
        -----
          G G E
          G A B
          -----
            F C F
            F B C
            -----
              A C G
                H J
              -----
                D H
```

Q1516 Find the greatest possible area of a quadrilateral having sides $2, 3, 4, 5$, in that order.

Q1517 A ball, here a point of zero dimension, lies on a "wedge–shaped billiard table" and continues to bounce off the sides as shown.



If the ball starts at a distance x_0 to the right and y_0 above the vertex of the wedge, and if the angle between its initial trajectory and the horizontal is θ, then find the closest distance the ball attains to the vertex.

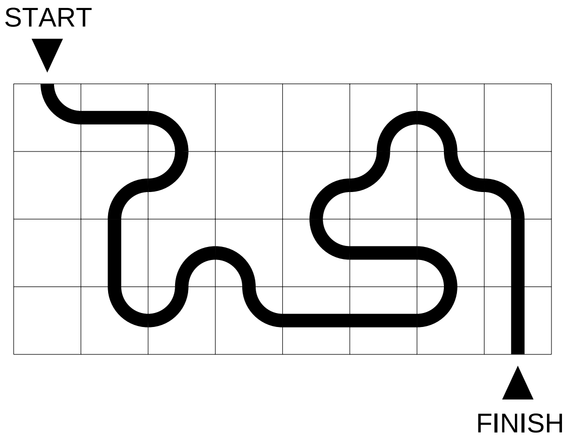
Q1518 Form a sequence of positive integers starting with 1, where each subsequent number is the smallest positive integer which cannot be written as the sum of four or fewer earlier numbers in the sequence, and where no earlier number is to be used more than once. Find the 2016th smallest number in the sequence.

Q1519 Suppose that $x_1, x_2, \ldots, x_n$ are n positive real numbers, with $n \geq 3$, and that $x_1 x_2 \cdots x_n = 1$. Prove that

$$\frac{1}{1 + x_1 + x_1 x_2} + \frac{1}{1 + x_2 + x_2 x_3} + \cdots + \frac{1}{1 + x_{n-1} + x_{n-1} x_n} + \frac{1}{1 + x_n + x_n x_1} > 1\,.$$

Q1520 This puzzle was inspired by the "Plumber Game" which can be found at `www.mathsisfun.com/games/plumber-game.html`.

A game is played on a 4×8 grid of squares. The aim is to create a path from START to FINISH by placing in some or all of the squares either a quarter–circle connection or a straight connection. An example of a successful path is shown.



Prove that a successful path must contain an odd number of straight connections.