Parabola Volume 52, Issue 3 (2016)

Solutions 1501–1510

Q1501 Find the sum of the sum of the sum of the digits for the number 2016^{2016}.

SOLUTION Write $S(n)$ for the sum of the digits of a positive integer n. We wish to find $S(S(S(n)))$ in the case when $n = 2016^{2016}$. Now $n < 10000^{2016} = 10^{8064}$, so n has at most 8064 digits; hence,

$$S(n) \le 9 \times 8064 < 100000\,.$$

Using the same idea again, this shows that $S(n)$ is a positive integer with at most 5 digits and so

$$S(S(n)) \le 9 \times 5 = 45\;;$$

Hence, $S(S(n))$ has at most two digits, with the "tens" digit no more than 4, so

$$S(S(S(n)) \le 4 + 9 = 13\,.$$

Note also that 2016 is a multiple of 9; therefore, so is n, and so is the sum of its digits $S(n)$, and so is the sum of *its* digits $S(S(n))$, and so is $S(S(S(n)))$. Therefore, $S(S(S(n)))$ is a positive multiple of 9 which is at most 13; the only possibility is $S(S(S(n))) = 9$.

Q1502 Recall the country described in Problem 1494: between every pair of towns there is a highway going in one direction but not in the other direction. As in the solution (see the previous issue), a town is called "central" if it can be reached from every other town either directly, or with just one intermediate town. If the number of towns in the country is odd, then show how to arrange the highways in such a way that **every** town is central.

SOLUTION Draw a schematic diagram where the towns (irrespective of their actual geographic locations) are arranged on a circle. From each town T, build a highway to every other town which is an odd number of steps around the circle from T in a clockwise direction, stopping just before we have gone all the way round the circle once. The first thing to do is to check that we have not infringed the highway-building rules by having a highway in both directions between the same pair of towns. If this were so for towns T_1 and T_2, then we would be able to travel along a highway from T_1 to T_2 and then another from T_2 to T_1; since each highway goes an odd number of steps around the circle, we have gone once around the circle from T_1 to T_1 in an even number of steps. But this is impossible as it takes n steps, an odd number, to go once round the circle.

So, we have a legitimate construction. Any town T can be reached directly from each town which is an odd number of steps anticlockwise from T; it can also be reached in two steps from any town an even number of steps anticlockwise from T by first going to the town before T on the circle. Therefore, every town is central.

NOW TRY Problem 1511.

Q1503 Use the method of Problem 1496 (solution in the previous issue) to solve the following problems.

(a) Find the maximum value of

$$\sqrt{1-x_1^2}+\sqrt{4-x_2^2}+\sqrt{9-x_3^2}\,,$$

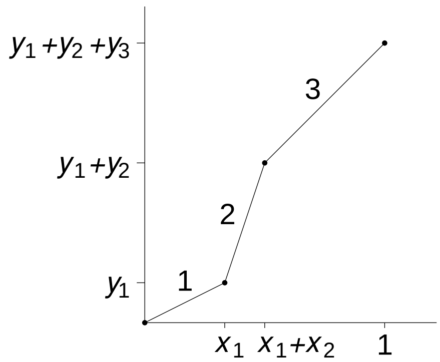
given that x_1, x_2, x_3 are positive real numbers with $x_1+x_2+x_3=1$.

(b) Find the minimum value of

$$\sqrt{(x-1)^2+(x^2-2)^2}+\sqrt{(x-3)^2+(x^2-4)^2}\,,$$

where x is a real number. Also, find the value of x which gives this minimum.

SOLUTION To solve (a), draw a diagram like that in the previous solution: the given expression can be written as $y_1+y_2+y_3$, where the distances from $(0,0)$ to (x_1,y_1) and then to (x_1+x_2, y_1+y_2) and then to $(x_1+x_2+x_3, y_1+y_2+y_3)$ are 1 and 2 and 3 respectively. So, we have to find the maximum height y for which $(1,y)$ can be reached from $(0,0)$ in steps of length $1,2,3$. The maximum will occur when we make these three steps collinear, forming a right triangle with side 1 and hypotenuse 6. The required maximum is the other side of the triangle, that is, $\sqrt{35}$.



For (b), the given expression represents the distance from $(1,2)$ to $(3,4)$ via the parabola $y=x^2$. Since $(1,2)$ is above the parabola and $(3,4)$ is below, the minimum length path will again be a straight line, and the distance is $2\sqrt{2}$. The value of x is obtained by finding the point (x,x^2) which is collinear with the given points:

$$\frac{x^2-2}{x-1}=\frac{4-2}{3-1}=1\,,$$

which simplifies to $x^2-x-1=0$ and hence

$$x=\frac{1+\sqrt{5}}{2}\ :$$

we have rejected the negative root since it is clear that for the intersection point we must have $1<x<3$.

NOW TRY Problem 1516.

Q1504 A sequence of numbers $a_1, a_2, a_3, \ldots$ is defined by the properties

$$a_1 = 3\,, \quad a_{n+1} = a_n^2 - 2 \text{ for } n \ge 1\,.$$

Find the value which

$$\frac{a_n}{a_1 a_2 a_3 \cdots a_{n-1}}$$

approaches as n becomes very large.

SOLUTION Let x be a number greater than 1 such that

$$a_1 = x + \frac{1}{x}$$

(we shall confirm later that such a number exists). Then

$$a_2 = \left(x + \frac{1}{x}\right)^2 - 2 = x^2 + \frac{1}{x^2}\,, \quad a_3 = \left(x^2 + \frac{1}{x^2}\right)^2 - 2 = x^4 + \frac{1}{x^4}$$

and so on: in general,

$$a_n = x^{2^{n-1}} + \frac{1}{x^{2^{n-1}}}\,.$$

Next, we have

$$\left(x - \frac{1}{x}\right)a_1 = \left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right) = x^2 - \frac{1}{x^2}$$
$$\left(x - \frac{1}{x}\right)a_1 a_2 = \left(x^2 - \frac{1}{x^2}\right)\left(x^2 + \frac{1}{x^2}\right) = x^4 - \frac{1}{x^4}$$

and in general

$$\left(x - \frac{1}{x}\right)a_1 a_2 a_3 \cdots a_{n-1} = x^{2^{n-1}} - \frac{1}{x^{2^{n-1}}}\,.$$

Therefore,

$$\frac{a_n}{a_1 a_2 a_3 \cdots a_{n-1}} = \frac{x^{2^{n-1}} + \dfrac{1}{x^{2^{n-1}}}}{x^{2^{n-1}} - \dfrac{1}{x^{2^{n-1}}}}\left(x - \frac{1}{x}\right) = \frac{1 + \dfrac{1}{x^{2^n}}}{1 - \dfrac{1}{x^{2^n}}}\left(x - \frac{1}{x}\right);$$

as n becomes very large the fraction $1/x^{2^n}$ becomes very small and the expression approaches

$$x - \frac{1}{x}\,.$$

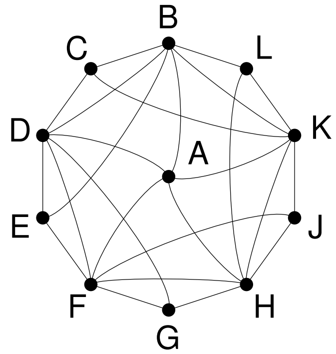
To finish, we have

$$x + \frac{1}{x} = 3 \quad \Leftrightarrow \quad x^2 - 3x + 1 = 0 \quad \Leftrightarrow \quad x = \frac{3 \pm \sqrt{5}}{2}\,;$$

in order to have $x > 1$ we choose the positive root, and hence,

$$\frac{a_n}{a_1 a_2 a_3 \cdots a_{n-1}} \to x - \frac{1}{x} = \frac{3 + \sqrt{5}}{2} - \frac{2}{3 + \sqrt{5}} = \sqrt{5}\,.$$

Q1505 It's easy to see that we can start at the point A in the following diagram, then travel along the lines in such a way as to visit every one of the labelled points without repeating any points: for example, $ABCDEFGHJKL$. But is it possible to do the same thing starting at B?



SOLUTION Starting at B, the task cannot be achieved. The required path must contain A, C, E, G, J, L, none of which can be accessed directly from any of the others. So these six points must be separated by the remaining five points B, D, F, H, K; but since B is to be the first point on the path, this is impossible.

Q1506 The following question is probably not all that hard, but see if you can solve it in three different ways: (a) by plain algebra; (b) by drawing a graph; (c) by calculus.

Let a, b, c be real numbers such that $a < b < c$. Show that the equation

$$\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} = 0$$

has one solution x between a and b, and another between b and c.

SOLUTION For the algebraic solution, it is clear that x cannot equal a, b or c, so the equation is equivalent to

$$(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b) = 0\,,$$

and expanding gives

$$3x^2 - (2a + 2b + 2c)x + (ab + ac + bc) = 0\,.$$

Write $f(x)$ for the left-hand side of this equation. Then

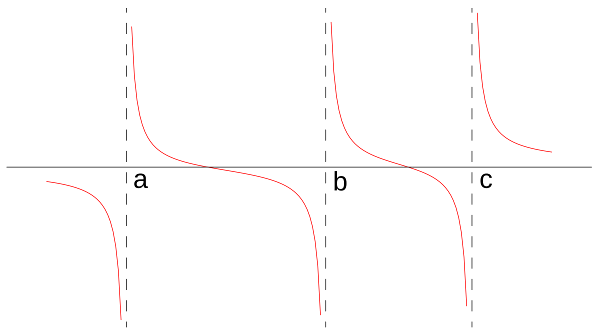
$$\begin{aligned} f(a) &= a^2 - ab - ac + bc = (a-b)(a-c)\,, \\ f(b) &= b^2 - ab - bc + ac = (b-a)(b-c)\,. \end{aligned}$$

Since $a < b < c$, we see that $f(a)$ is positive and $f(b)$ is negative; so the quadratic has a solution between a and b. Similarly (try it yourself!) we can show that $f(c)$ is positive, so there is another solution between b and c.

For a graphical solution, sketch the curve

$$y = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}\,.$$

If x is slightly greater than c, then $x - c$ is a very small positive number, so $\frac{1}{x-c}$ is very large. On the other hand, if x is slightly less than c, then $x - c$ is a very small negative number, so $\frac{1}{x-c}$ is very large in the negative direction. Thinking along similar lines for b and a shows that the graph must look like this.



The graph is an unbroken curve, except at the points a, b, c, so there is a solution between a and b, and another between b and c. Moreover, the quantities $\frac{1}{x-a}$ and $\frac{1}{x-b}$ and $\frac{1}{x-c}$ are always decreasing between the points a and b and c, so there is only one solution in each interval.

For a calculus solution, pick up the algebraic solution at the point where we had

$$f(x) = 3x^2 - (2a + 2b + 2c)x + (ab + ac + bc) = 0\,,$$

and note that $f(x)$ is the derivative of

$$g(x) = (x - a)(x - b)(x - c)\,.$$

The cubic has roots at a and b, and therefore has a turning point between a and b. At this turning point the derivative is zero, that is, $f(x) = 0$, so there is a solution between a and b; and, likewise, another between b and c.

Q1507 Form a six-digit number from a standard keypad

1	2	3
4	5	6
7	8	9

by going along any row, column, or diagonal and then back again. For example, we could get 258852 or 951159, among others. Prove that every number which can be obtained in this way is a multiple of 1221.

SOLUTION Each row, column or diagonal forms an arithmetic sequence $a,\ a+d,\ a+2d$. (For example, a column going from top to bottom has $d=3$, while the diagonal from bottom right to top left has $d=-4$.) Therefore, the numbers we can obtain have digits

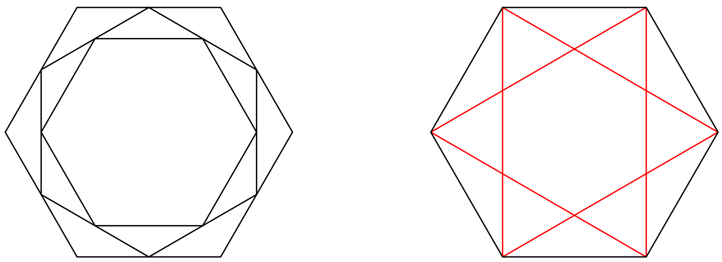
$$a,\ a+d,\ a+2d,\ a+2d,\ a+d,\ a$$

and we can write

$$\begin{aligned} n &= 100000a + 10000(a+d) + 1000(a+2d) + 100(a+2d) + 10(a+d) + a \\ &= 111111a + 12210d \\ &= 1221(91a+10d)\,, \end{aligned}$$

which is a multiple of 1221.

Q1508 In the left-hand diagram we begin with a regular hexagon and join the midpoints of the sides to form a smaller regular hexagon; we then repeat the same procedure a second time to obtain a still smaller hexagon; we could repeat this step as many times as we wish.



In the right-hand diagram we begin with another hexagon, of the same size as the original hexagon on the left, and join every second vertex so that a smaller hexagon is formed inside the original. We only do this once. How many times must we perform the "midpoint" construction so that the resulting hexagon will lie inside the small hexagon in the right-hand diagram?

SOLUTION Let s_n be the side length of the hexagon after n steps of the midpoint construction. (So, s_0 is the side length of the original hexagon.) By the cosine rule,

$$s_{n+1}^2 = \left(\frac{s_n}{2}\right)^2 + \left(\frac{s_n}{2}\right)^2 - 2\left(\frac{s_n}{2}\right)\left(\frac{s_n}{2}\right)\cos 120^\circ = \frac{3}{4}s_n^2$$

and so

$$s_{n+1} = \frac{\sqrt{3}}{2}s_n\,.$$

Let t be the side of the smaller hexagon in the right-hand diagram. We can find t by using the cosine rule again, or more simply, observe that

$$3t = 2s_1 = \sqrt{3}\,s_0 \quad \text{and so} \quad t = \frac{1}{\sqrt{3}}s_0\,.$$

To make the side length in the left-hand diagram after n steps less than t we require

$$s_n \le t \quad \Leftrightarrow \quad \left(\frac{\sqrt{3}}{2}\right)^n s_0 \le \frac{1}{\sqrt{3}} s_0 \quad \Leftrightarrow \quad \left(\frac{3}{4}\right)^n \le \frac{1}{3},$$

and a quick calculation shows that this occurs for the first time when $n = 4$. However, since 4 is even, the left-hand hexagon obtained after 4 steps will have a different orientation from that on the right: even though it is smaller, it will stick out beyond the right-hand hexagon. So we need to go one step further, and the answer is that we must perform the midpoint construction 5 times.

Q1509 If $0 < \theta < \frac{1}{2}\pi$, show that

$$\log_{1+\sin\theta}(\cos\theta) + \log_{1-\sin\theta}(\cos\theta) = 2\log_{1+\sin\theta}(\cos\theta)\log_{1-\sin\theta}(\cos\theta).$$

SOLUTION Note that since $0 < \theta < \frac{1}{2}\pi$, we have $0 < \cos\theta < 1$ and $0 < \sin\theta < 1$. Starting with simple algebra and then using properties of logarithms,

$$\begin{aligned}\frac{\log_{1+\sin\theta}(\cos\theta) + \log_{1-\sin\theta}(\cos\theta)}{\log_{1+\sin\theta}(\cos\theta)\log_{1-\sin\theta}(\cos\theta)} &= \frac{1}{\log_{1-\sin\theta}(\cos\theta)} + \frac{1}{\log_{1+\sin\theta}(\cos\theta)} \\ &= \log_{\cos\theta}(1-\sin\theta) + \log_{\cos\theta}(1+\sin\theta) \\ &= \log_{\cos\theta}(1-\sin^2\theta) \\ &= \log_{\cos\theta}(\cos^2\theta) \\ &= 2.\end{aligned}$$

Q1510 If n and r are positive integers, then the set $S = \{1, 2, 3, \ldots, n\}$ has $\binom{n}{r}$ subsets of size r. Here, $\binom{n}{r}$ is the binomial coefficient, sometimes written as $C(n, r)$ or ${}^nC_r$.

(a) How many subsets of S have size r and largest element k?

(b) Prove that

$$\binom{r-1}{r-1} + \binom{r}{r-1} + \binom{r+1}{r-1} + \cdots + \binom{n-1}{r-1} = \binom{n}{r}.$$

(c) Show that

$$k\binom{m}{k} = m\binom{m-1}{k-1}.$$

(d) If from every size r subset of S we write down the largest element, what is the total of the numbers we have written? (Hint: simplify your expression by using the results of (b) and (c).)

SOLUTION Clearly the largest element in a size r subset of the set $\{1, 2, 3, \ldots, n\}$ must be at least r. So if $k < r$, then the answer to (a) is zero. If the largest element is $k \ge r$, then the subset must also contain $r-1$ elements chosen from $\{1, 2, \ldots, k-1\}$, and the number of possibilities is

$$\binom{k-1}{r-1}.$$

For (b), consider the problem of selecting a size r subset of S by first choosing its largest element. This largest element must be

$$r \quad \text{or} \quad r+1 \quad \text{or} \quad r+2 \quad \text{or} \quad \cdots \quad \text{or} \quad n\,;$$

from (a), the number of subsets is respectively

$$\binom{r-1}{r-1} \quad \text{or} \quad \binom{r}{r-1} \quad \text{or} \quad \binom{r+1}{r-1} \quad \text{or} \quad \cdots \quad \text{or} \quad \binom{n-1}{r-1}\,;$$

if we add these expressions we get the total number of size r subsets of S, and this proves the stated equation.

Next, consider the problem of choosing a size k subset of $\{1, 2, \ldots, m\}$ and then choosing one particular element of that subset. Doing this in the obvious way, there are $\binom{m}{k}$ ways to choose the set and then k to choose the element,

$$k\binom{m}{k}$$

options altogether. On the other hand, we could first choose any element of $\{1, 2, \ldots, m\}$ as the "particular" element, and then complete the subset by choosing $k-1$ from the $m-1$ remaining elements. The number of ways of doing this is

$$m\binom{m-1}{k-1}.$$

Since these two expressions are different ways of writing the answer to the same problem, they must be equal: this answers (c).

From our solution to (b), we can see that in our collection of largest elements, r will occur $\binom{r-1}{r-1}$ times, $r+1$ will occur $\binom{r}{r-1}$ times, and so on. So the total of all the largest elements is

$$T = r\binom{r-1}{r-1} + (r+1)\binom{r}{r-1} + (r+2)\binom{r+1}{r-1} + \cdots + n\binom{n-1}{r-1}\,;$$

we shall simplify this by using the results of (b) and (c). For the first term, apply (c) with $m = r$, $k = r$; for the second, apply (c) with $m = r+1$, $k = r$; and so on. This gives

$$T = r\binom{r}{r} + r\binom{r+1}{r} + r\binom{r+2}{r} + \cdots + r\binom{n}{r}.$$

If we take out the common factor of r, then what remains is the left-hand side of (b), replacing n by $n+1$ and r by $r+1$. So, our final simplified answer is

$$T = r\binom{n+1}{r+1}.$$