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An exploration of pandivisible numbers

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Our system for writing integers relies on ten symbols. When we write an integer that is less than ten, the rule is easy: write the corresponding symbol; for example, we all agree than "nine" should be expressed as "9". However, for integers greater than or equal to ten, the rules are more complicated. To write the number "twenty-three", we combine the symbols for two and three in a specific order so that we write "23". But what does this really mean? Well, our number system is based on powers of ten. When we write 23, what we really mean is $2 \times 10^1 + 3 \times 10^0$. Because of this, we call the number system we use "base 10". This immediately raises the question of what would happen if we used other numbers as bases. A condition usually imposed on base systems is that digits in base n can only come from the set $\{0, 1, 2, \ldots, n-1\}$. For example, 123 in base 9 (which is usually written 123_9) is

$$123_9 = 1 \times 9^2 + 2 \times 9^1 + 3 \times 9^0 = 102_{10}\,.$$

Now that we have a generalized system for writing numbers, we can ask some interesting questions. For example, a *pandigital* number is one that uses each numeral from 1 to $b-1$ exactly once. How many pandigital numbers are there in base b? Well, we have $b-1$ possible numerals for the first digit, and then $b-2$ possible numerals for the second, and so on, so there are $(b-1)!$ possible pandigital numbers in base b. This isn't exactly a result that is difficult to prove, but it is nevertheless an interesting result, and we haven't had to do much work in order to achieve it.

We can now define another interesting term. A number is called *polydivisible* in base b if the first two digits, taken as a number in base b, are divisible by two, the first three digits taken as a number in base b are divisible by three, and so on. For example, you can easily check that 4232_6 is a polydivisible number. These classes become very interesting when we combine them.

A *pandigital polydivisible* number is one that satisfies the criteria for both pandigital and polydivisible numbers. To reduce wordiness, we will hereafter call these *pandivisible* numbers. Our first observation is that there must be exactly $b-1$ digits in a base b pandivisible number; otherwise, it would not be pandigital. But then we come to a surprising result: there are no pandivisible numbers in any odd base. We can prove this by contradiction.

Suppose that in base $2n+1$, there exists a pandivisible number $k_1k_2\ldots k_{2n}$, where each k_i represents a distinct digit. Then $2n \mid k_1k_2\ldots k_{2n}$. This means that $k_1k_2\ldots k_{2n}$ leaves a remainder of 0 when divided by $2n$. That is,

$$k_1k_2\ldots k_{2n} \equiv 0 \pmod{2n}\,.$$

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We can expand the left side, as we did for 123_9 above:

$$(2n+1)^{2n-1}k_1 + (2n+1)^{2n-2}k_2 + \cdots + (2n+1)^0 k_{2n} \equiv 0 \pmod{2n}. \tag{1}$$

For any natural number a, the Binomial Theorem states that

$$(2n+1)^a = \binom{a}{0}(2n)^0 + \binom{a}{1}(2n)^1 + \binom{a}{2}(2n)^2 + \cdots + \binom{a}{a}(2n)^a,$$

and we can see that this leaves a remainder of 1 upon division by $2n$. This simplifies (1) dramatically. It now becomes

$$k_1 + k_2 + \cdots + k_{2n} \equiv 0 \pmod{2n}. \tag{2}$$

Because this number is pandigital, every one of the digits must be a distinct number from 1 to $2n$, and each of these must be used, in some order. Thus, we see that (2) can be simplified again to

$$1 + 2 + 3 + \cdots + (2n-1) + 2n \equiv 0 \pmod{2n}. \tag{3}$$

We see that the left side of (3) is the sum of the first $2n$ natural numbers, so we finally have

$$\frac{2n(2n+1)}{2} = n(2n+1) \equiv 0 \pmod{2n}. \tag{4}$$

However, notice that $2n+1$ and $2n$ are relatively prime. Additionally, n is not a multiple of $2n$. This means that $n(2n+1)$ is not a multiple of $2n$, and so (4) cannot be true, which means that our assumption has led to a contradiction. Hence, there are no pandivisible numbers in any odd base.

From here on, we assume that the base is even. Using the notation above, where k_m is the mth digit from the left of a pandivisible number in base b, we can see that m and k_m must have the same parity. For $k_1k_2\ldots k_m$ to be divisible by an even m, k_m must be even. This means that if m is odd, then k_m must be odd as well, since all the even numbers are already taken. Now we see that all pandivisible numbers must have digits that alternate parity.

In fact, we can generalize this observation, to see that if $k_1k_2\ldots k_{b-1}$ is a pandivisible number in base b, then $\gcd(m,b) \mid k_m$. To see this, we again take the number $k_1k_2\ldots k_m$ in base b. Now, for the number to be pandivisible, it must be that

$$k_1k_2k_3\ldots k_m \equiv 0 \pmod{m}.$$

This expands, as before, to become

$$b^{m-1}k_1 + b^{m-2}k_2 + \cdots + b^1k_{m-1} + b^0k_m \equiv 0 \pmod{m}.$$

Since $\gcd(m,b) \mid m$, we know that

$$b^{m-1}k_1 + b^{m-2}k_2 + \cdots + b^1k_{m-1} + b^0k_m \equiv 0 \pmod{\gcd(m,b)}.$$

But $\gcd(m, b)$ also divides b, so it also divides all perfect powers of b. Thus

$$k_m \equiv 0 \pmod{\gcd(m, b)},$$

which implies that $\gcd(m, b) \mid k_m$, as desired. This has the interesting consequence that if $m \mid b$, then $m \mid k_m$. This follows because if $m \mid b$, then $\gcd(m, b) = m$. A particular application of this consequence occurs if we let $m = \frac{b}{2}$. Note that m is an integer because b is even. In this case, $\frac{b}{2} \mid k_{b/2}$, but we also know that $k_{b/2} < b$. There is only one positive integer that satisfies both criteria: we must have $k_{b/2} = \frac{b}{2}$.

We can use these facts to search for pandivisible numbers. Obviously, if we just try an exhaustive search through all pandigital numbers in base b, then there would be $(b-1)!$ numbers to test. But we can immediately narrow our search, since we know that $k_{b/2} = \frac{b}{2}$. If we searched through all pandigital numbers with this property, then we would still test $(b-2)!$ numbers. We could instead use the property that the digits of pandivisible numbers alternate parity. There are $\left(\frac{b}{2}\right)!$ arrangements of odd digits and $\left(\frac{b}{2}-1\right)!$ arrangements of even digits, so by applying the rule that digits alternate parity, we only have to test $\frac{b}{2}\left(\left(\frac{b}{2}-1\right)!\right)^2$ numbers.

To optimize the search further, we can combine these two rules. In the case where $b = 4n + 2$, it must hold that $\frac{b}{2}$ is odd. Thus, there are $\left(\frac{b}{2}-1\right)!$ ways to arrange the remaining odd digits and $\left(\frac{b}{2}-1\right)!$ ways to arrange the even digits, so $\left(\left(\frac{b}{2}-1\right)!\right)^2$ numbers must be tested. If $b = 4n$, then $\frac{b}{2}$ is even, so there are $\left(\frac{b}{2}\right)!$ ways to arrange the odd digits and $\left(\frac{b}{2}-2\right)!$ ways to arrange the remaining even digits, so $\left(\frac{b}{2}\right)!\left(\frac{b}{2}-2\right)!$ numbers must be tested. Of course, for specific values of b, the search can be optimized even further by using the rule that $\gcd(m, b) \mid k_m$. For example, in base 12, this rule implies that $3 \mid k_3, k_6, k_9$. But $k_6 = 6$, since 6 is half the base, so we now know either that $k_3 = 3$ and $k_9 = 9$ or that $k_3 = 9$ and $k_9 = 3$ in any possible base 12 pandivisible number.

There is much about these numbers that is yet to be discovered. For example, an even base does not guarantee the existence of a pandivisible number: none exist in base 12. (The Appendix contains a table of the pandivisible numbers in selected bases.) This raises the question of whether there are an infinite number of pandivisible numbers. Is there more hidden structure in these numbers? Additionally, what happens if we change the conditions? Maybe we allow -1 as a digit, or perhaps we only allow the digits to be odd. As you can see, this is a problem that is ripe for investigation, and there appears to be much to explore.

Appendix

The following table lists all the pandivisible numbers in the even bases from 2 to 18, inclusive. The odd bases have been omitted because no pandivisible numbers exist in those bases, as we have proven.

Base	Pandivisible Numbers
2	1
4	123, 321
6	14325, 54321
8	3254167, 5234761, 5674321
10	381654729
12	None
14	9C3A5476B812D
16	None
18	None