Parabola Volume 53, Issue 1 (2017)

Solutions 1511–1520

Q1511 In a certain country (see Problems 1494 and 1502), every pair of towns is connected by a highway going in one direction but not by a highway going in the other direction. A town is called "central" if it can be reached from every other town either directly, or with just one intermediate town.

(a) Show that if there are 8 towns in this country, then it is possible for *every* town to be central.

(b) Show that the same is true for any number of towns except 2 or 4.

SOLUTION

(a) With a bit of trial and error, we see that the following will work:

from	1	2	3	4	5	6	7	8
to	2,4,5	3,5,8	1,4,6,7	2,5,6,7	3,6,8	1,2,7	1,2,5,8	1,3,4,6

You may easily check that this is a legitimate arrangement (there is never a highway both from x to y and from y to x) and that every town is accessible from every other town in either one or two steps.

(b) We shall show that if it is possible to arrange highways in a country of n towns (with $n > 1$) in such a way that every town is central, then the same is true in a country of $2n$ towns.

Suppose that we have a suitable arrangement for a country with towns $T_1, \ldots, T_n$. Consider a country with $2n$ towns; suppose that it is divided into two states called A and B, and that the towns are called $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$, respectively. (The people who choose town names in these countries are not very imaginative.) Now construct highways as follows:

- highways in A are arranged as in the smaller country: that is, there is a highway from A_i to A_j if and only if there is a highway from T_i to T_j;
- highways in B are arranged as in the smaller country but with directions reversed: that is, there is a highway from B_i to B_j if and only if there is a highway from T_j to T_i;
- the highway between A_i and B_j will be directed from A_i to B_j if $i = j$, and from B_j to A_i if $i \neq j$.

Since every town in the smaller country is central, it is possible to get from any town in A to any other town in A with at most one intermediate town; this property is not changed by reversing the directions of all highways, so the same is true in B. We have to check that we can get to every town in B from every town in A with at most one intermediate town, and likewise to A from B.

So, consider any town B_j. If $i = j$, then there is a direct connection $A_i \to B_j$. If $i \neq j$ and there is a highway $B_i \to B_j$, then we have $A_i \to B_i \to B_j$. If $i \neq j$ and there is no highway $B_i \to B_j$, then $B_j \to B_i$ (remember that *every* pair of towns is connected one way or the other), so $T_i \to T_j$; hence, $A_i \to A_j$ and we have $A_i \to A_j \to B_j$. Thus, B_j can be reached from every A_i in at most two steps.

Finally, consider any town A_i. If $j \neq i$, then we have $B_j \to A_i$. If $j = i$, then choose a town T_k having a highway to T_i. (There must be such a town, otherwise T_i would be inaccessible and hence not central: this is why we required $n > 1$.) Therefore, $A_k \to A_i$ and, since $k \neq j$, we have $B_j \to A_k \to A_i$.

This shows that every town is central in the country of $2n$ towns. Since we know that it is possible to make every town central in a country of 8 towns (above) or $1, 3, 5, 7, \ldots$ towns (Problem 1502, solution in last issue), doubling a sufficient number of times (but not for $n = 1$) gives an arrangement in which every town is central for any number of towns except 2 or 4.

Q1512 Find all solutions of the simultaneous equations

$$x^2 + 4y^2 + z^2 = 20 \quad \text{and} \quad x + yz = 6\,.$$

SOLUTION Substituting $x = 6 - yz$ into the first equation, expanding and collecting the terms in z gives

$$(y^2 + 1)z^2 - 12yz + (4y^2 + 16) = 0\,. \tag{$*$}$$

In order to obtain (real) solutions for z, this quadratic equation must have a non–negative discriminant

$$(12y)^2 - 4(y^2 + 1)(4y^2 + 16) \geq 0\,.$$

A bit more algebra leads to

$$-(y^2 - 2)^2 \geq 0\,;$$

since a square cannot be negative, the only possibility is $y^2 = 2$. Substituting $y = \sqrt{2}$ back into $(*)$ gives

$$3z^2 - 12\sqrt{2}\,z + 24 = 0$$

which factorises as $3(z - 2\sqrt{2})^2 = 0$. This provides the value of z and we calculate $x = 6 - yz$, giving the solution

$$x = 2\,, \quad y = \sqrt{2}\,, \quad z = 2\sqrt{2}\,.$$

Setting $y = -\sqrt{2}$ and following the same procedure yields the only other solution

$$x = 2\,, \quad y = -\sqrt{2}\,, \quad z = -2\sqrt{2}\,.$$

Q1513 Divide the following array of numbers

2	2	1	2	1	1	1
2	4	3	1	1	2	2
1	1	2	1	1	2	1
5	1	3	1	6	1	1
1	3	3	1	1	1	4

into 11 connected regions, each containing numbers adding up to 6. A “connected region” means a set of squares in which every square is joined to some other square along an edge (not just a corner). No combination of numbers in a region may be used more than once. For example, you might use one of the following regions:

1	1	2	2

1	2
2	1

1	1
	2
	2

However, you may not use more than one of them because they all contain the same numbers.

SOLUTION First, we note that there are exactly 11 collections of positive integers which add up to 6, namely,

$$\begin{aligned} 6 &= 5+1 \\ &= 4+2 = 4+1+1 \\ &= 3+3 \\ &= 3+2+1 \\ &= 3+1+1+1 \\ &= 2+2+2 \\ &= 2+2+1+1 \\ &= 2+1+1+1+1 \\ &= 1+1+1+1+1+1\,. \end{aligned}$$

Therefore, we must use every one of these once each. Obviously, the 6 must go in a region by itself. There are two options for $2+2+2$ but if we take that in the top left corner, then there is nowhere to obtain $4+2$. The rest can be done by trial and error.

2	2	1	2	1	1	1
2	4	3	1	1	2	2
1	1	2	1	1	2	1
5	1	3	1	6	1	1
1	3	3	1	1	1	4

Q1514 Find two (non-constant) polynomials whose product is the polynomial

$$f(x) = 1 + x^{1010} + x^{1011} + x^{1012} + x^{1013} + \cdots + x^{2017} .$$

SOLUTION We have

$$\begin{aligned} f(x) &= (1 + x + x^2 + \cdots + x^{2017}) - (x + x^2 + x^3 + \cdots + x^{1009}) \\ &= (1 + x^{1009})(1 + x + x^2 + \cdots + x^{1008}) - x(1 + x + x^2 + \cdots + x^{1008}) \\ &= (1 - x + x^{1009})(1 + x + x^2 + \cdots + x^{1008}) , \end{aligned}$$

which shows the two required polynomials.

Q1515 Replace each letter by a different digit in the following long division in such a way that the working is correct. You may assume (as stated in the article by Prof Miklos N. Szilagyi earlier this issue) that no number begins with a zero.

$$\begin{array}{r|ccccccc}
 & & & \mathrm{A} & \mathrm{C} & \mathrm{H} & \mathrm{G} & \mathrm{A} \\ \cline{2-8}
\mathrm{H\,J} & \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} & \mathrm{F} & \mathrm{G} \\
 & & \mathrm{H} & \mathrm{J} & & & & \\ \cline{2-4}
 & & \mathrm{C} & \mathrm{D} & \mathrm{D} & & & \\
 & & \mathrm{B} & \mathrm{F} & \mathrm{G} & & & \\ \cline{3-5}
 & & & \mathrm{G} & \mathrm{G} & \mathrm{E} & & \\
 & & & \mathrm{G} & \mathrm{A} & \mathrm{B} & & \\ \cline{4-6}
 & & & & \mathrm{F} & \mathrm{C} & \mathrm{F} & \\
 & & & & \mathrm{F} & \mathrm{B} & \mathrm{C} & \\ \cline{5-7}
 & & & & & \mathrm{A} & \mathrm{C} & \mathrm{G} \\
 & & & & & & \mathrm{H} & \mathrm{J} \\ \cline{6-8}
 & & & & & & \mathrm{D} & \mathrm{H}
\end{array}$$

SOLUTION Because a number cannot begin with zero, A, B, C, D, F, G, H cannot be zero. Since the first multiple (and the last) is equal to the divisor HJ, we have $A = 1$. Next, the subtraction $ABC - HJ = CD$ in lines 2–4 shows that $B < H$ as, otherwise, the difference would have three digits; the subtraction in lines 10–12 shows in the same way that $C < H$. The maximum possible multiple of HJ is

$$9(HJ) = 9(10H + J) < 100(H + 1) ;$$

so any multiple of HJ has its hundreds digit at most H. Looking at lines 5, 7, 9 shows that $B, G, F < H$.

We now know that A, B, C, D, F, G, H are all different and not zero, and H is the largest of them: so $H \geq 7$. Now the subtraction in lines 10–12 is

$$ACG - HJ = DH ;$$

this shows that G is the final digit of $H + J$, which we write as $G \equiv H + J$. (If you have studied modular arithmetic, then you will recognise that this is really congruence modulo 10.) Similar ideas for other subtractions yield $D \equiv 2G$ and $C \equiv D + J$ and $F \equiv 2C$ and $E \equiv B + C$. The rest of the solution will be by trial and error. If we try $H = 7$, then we know from above that $J = 8, 9$ or 0. We have

H	J	G	D	C
7	8	5	0	8
7	9	6	2	1
7	0	7	4	4

But each of these cases is impossible, either because there is a repeated digit or because there is a letter (other than A) which represents 1. Similar ideas eliminate all possibilities except

H	J	G	D	C
8	9	7	4	3
9	3	2	4	7
9	4	3	6	0
9	5	4	8	3
9	8	7	4	2

Finally, lines 4 and 5 show that $B < C$; this, together with $E \equiv B + C$, leaves only the first line as a possibility. So, we are left with the unique solution

$$H = 8\,, \quad J = 9\,, \quad G = 7\,, \quad D = 4\,, \quad C = 3\,, \quad F = 6\,, \quad E = 5\,, \quad B = 2\,,$$

and the reconstructed division is

$$\begin{array}{r|ccccccc}
 & & & 1 & 3 & 8 & 7 & 1 \\ \cline{2-8}
8\,9 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
 & & 8 & 9 & & & & \\ \cline{2-4}
 & & 3 & 4 & 4 & & & \\
 & & 2 & 6 & 7 & & & \\ \cline{3-5}
 & & & 7 & 7 & 5 & & \\
 & & & 7 & 1 & 2 & & \\ \cline{4-6}
 & & & & 6 & 3 & 6 & \\
 & & & & 6 & 2 & 3 & \\ \cline{5-7}
 & & & & & 1 & 3 & 7 \\
 & & & & & & 8 & 9 \\ \cline{6-8}
 & & & & & & 4 & 8
\end{array}$$

Q1516 Find the greatest possible area of a quadrilateral having sides $2, 3, 4, 5$, in that order.

SOLUTION Draw a diagonal of length $\sqrt{t}$ to form triangles with sides $2, 3, \sqrt{t}$ and $4, 5, \sqrt{t}$. Using Heron's Formula for the area of a triangle in terms of its sides, the first triangle has area

$$\begin{aligned} 4A &= \sqrt{\bigl(2+3+\sqrt{t}\bigr)\bigl(2+3-\sqrt{t}\bigr)\bigl(2+\sqrt{t}-3\bigr)\bigl(3+\sqrt{t}-2\bigr)} \\ &= \sqrt{(25-t)(t-1)} \\ &= \sqrt{-25+26t-t^2} \\ &= \sqrt{12^2-(t-13)^2}\,. \end{aligned}$$

Doing the same sort of thing for the other triangle, the total area of the quadrilateral is given by

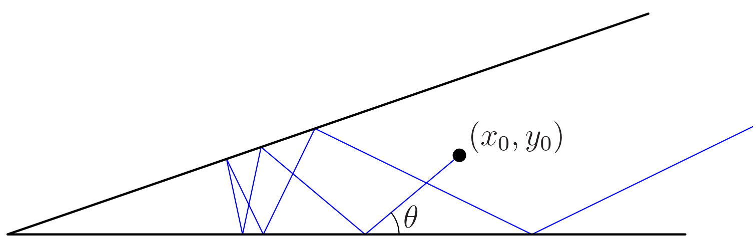
$$4A = \sqrt{12^2-(t-13)^2} + \sqrt{40^2-(41-t)^2}\,. \qquad (*)$$

To maximise this area, we use the technique of Problem 1503(a) – see the solution in the previous issue. The right hand side of $(*)$ is the vertical distance gained by the broken line going from $(13, 0)$ to (t, y_1) and then to $(41, y_1 + y_2)$, where the two line segments have lengths 12 and 40. We maximise the height gain by making the two parts collinear, giving a right–angled triangle with hypotenuse 52, horizontal side length $41 - 13 = 28$ and therefore vertical side length

$$\sqrt{52^2-28^2} = \sqrt{80\times 24} = 8\sqrt{30}\,.$$

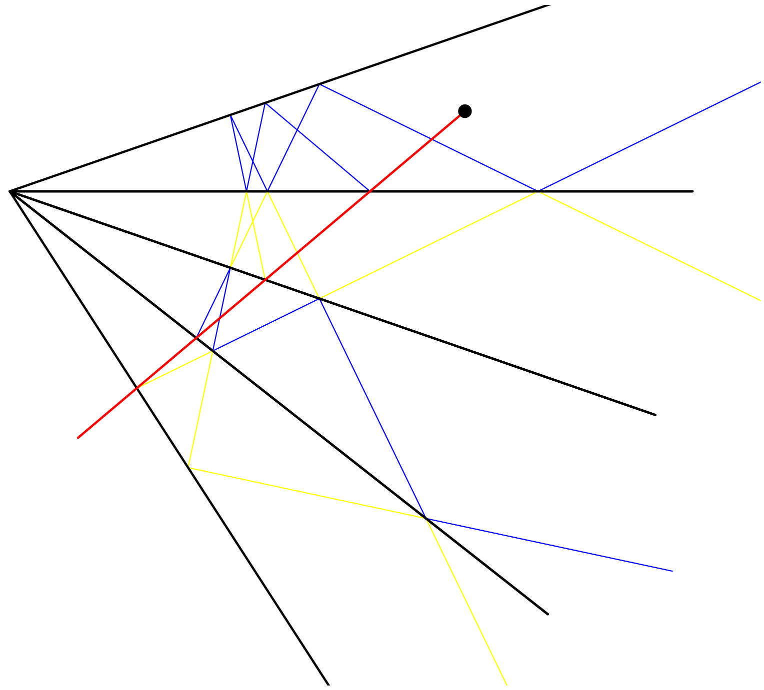
Therefore, the maximum area is $2\sqrt{30}$.

Q1517 A ball (which can be thought of as a point of zero dimension) is projected into a "wedge–shaped billiard table" and continues to bounce off the sides as shown.



If the ball starts at a distance x_0 to the right and y_0 above the vertex of the wedge, and if the angle between its initial trajectory and the horizontal is θ, then find the closest distance the ball attains to the vertex.

SOLUTION Imagine that instead of the wedge remaining fixed and the ball bouncing, the ball keeps going and the wedge is reflected; and that this happens every time the ball hits a wall. The scenario looks like this:



Since, for the bouncing ball (ignoring any spin), the angle of incidence equals the angle of reflection, the successive segments of the ball's path form a straight line (shown in red in the diagram). So, the answer that we are looking for is the shortest distance from this line to the vertex. If we treat the vertex as the origin in the Cartesian plane, then the line goes through (x_0, y_0) and has gradient $\tan\theta$; by a well–known formula, the distance from the line to the origin is

$$|x_0 \sin\theta - y_0 \cos\theta| \, .$$

NOW TRY Problem 1527.

Q1518 Form a sequence of positive integers starting with 1, where each subsequent number is the smallest positive integer which cannot be written as the sum of four or fewer earlier numbers in the sequence, no earlier number to be used more than once. Find the 2016th smallest number in the sequence.

SOLUTION The sequence is

$$1\,,\quad 2\,,\quad 4\,,\quad 8\,,\quad 16\,,\quad 31\,,\quad 46\,,\quad 61\,,\quad 76\,,\ldots\,:$$

that is, powers of 2 up to 16, and every 15th number thereafter. To prove this, it is easy to check that the sequence starts $1, 2, 4, 8, 16$. From now on, if $n = 16 + 15k$ is the last number in the sequence so far, then

$$n+1\,,\quad n+2\,,\ldots,\quad n+14$$

can all be written as the sum of n together with three or fewer of the numbers $1, 2, 4, 8$, and therefore are not in the sequence. Consider $m = n + 15$ and suppose it can be written as a sum of at most four previous terms. Note that m divided by 15 leaves remainder 1; and the previous terms divided by 15 have remainders $2, 4, 8$ once each, and 1 lots of times. By trial and error, the only way to add up four or fewer of these and get a remainder of 1 is to take a single term with remainder 1. That is, m is equal to one of the previous terms in the sequence; but this is clearly not the case. Therefore, m cannot be written as a sum of four earlier terms, and so it is the next term in the sequence. Thus, as claimed, the sequence after 16 consists of every 15th term.

To answer the question: 16 is the 5th term, so the 2016th is 2011 terms further along and it is

$$16 + 2011 \times 15 = 30181 \,.$$

Q1519 Suppose that $x_1, x_2, \ldots, x_n$ are n positive real numbers, with $n \geq 3$, and that $x_1 x_2 \cdots x_n = 1$. Prove that

$$\frac{1}{1 + x_1 + x_1x_2} + \frac{1}{1 + x_2 + x_2x_3} + \cdots + \frac{1}{1 + x_{n-1} + x_{n-1}x_n} + \frac{1}{1 + x_n + x_nx_1} > 1 \,.$$

SOLUTION Consider the positive numbers

$$y_1 = 1 \,, \quad y_2 = x_1 \,, \quad y_3 = x_1x_2 \,, \quad y_4 = x_1x_2x_3$$

and so on, finishing with

$$y_n = x_1x_2 \cdots x_{n-1} \quad \text{and} \quad y_{n+1} = x_1x_2 \cdots x_n \,.$$

Note that

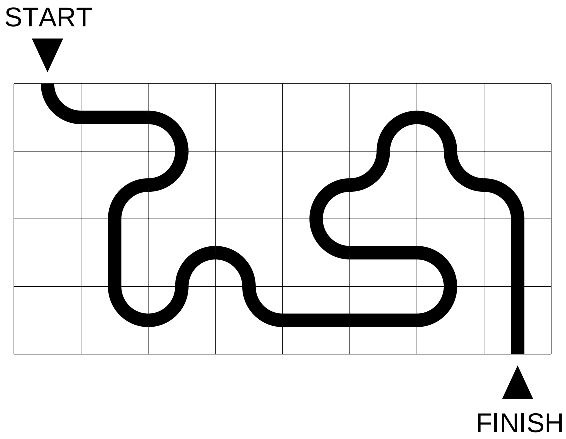
$$x_k = \frac{y_{k+1}}{y_k}$$

for each k, and that $y_{n+1} = 1 = y_1$. We have

$$\begin{aligned}
&\frac{1}{1 + x_1 + x_1x_2} + \frac{1}{1 + x_2 + x_2x_3} + \cdots + \frac{1}{1 + x_n + x_nx_1} \\
&\quad = \frac{1}{1 + \frac{y_2}{y_1} + \frac{y_2}{y_1}\frac{y_3}{y_2}} + \frac{1}{1 + \frac{y_3}{y_2} + \frac{y_3}{y_2}\frac{y_4}{y_3}} + \cdots + \frac{1}{1 + \frac{y_{n+1}}{y_n} + \frac{y_{n+1}}{y_n}\frac{y_2}{y_1}} \\
&\quad = \frac{y_1}{y_1 + y_2 + y_3} + \frac{y_2}{y_2 + y_3 + y_4} + \cdots + \frac{y_n}{y_n + y_1 + y_2} \\
&\quad > \frac{y_1}{y_1 + y_2 + \cdots + y_n} + \frac{y_2}{y_1 + y_2 + \cdots + y_n} + \cdots + \frac{y_n}{y_1 + y_2 + \cdots + y_n} \\
&\quad = \frac{y_1 + y_2 + \cdots + y_n}{y_1 + y_2 + \cdots + y_n} \\
&\quad = 1 \,.
\end{aligned}$$

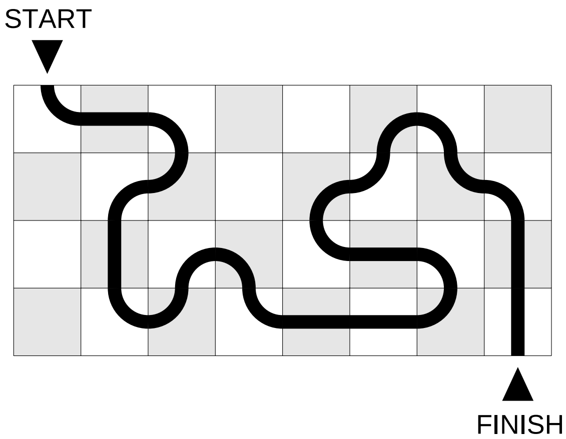
Q1520 This puzzle was inspired by the "Plumber Game" which can be found at `www.mathsisfun.com/games/plumber-game.html`.

A game is played on a 4×8 grid of squares. The aim is to create a path from START to FINISH by placing in some or all of the squares either a quarter–circle connection or a straight connection. An example of a successful path is shown.



Prove that a successful path must contain an odd number of straight connections.

SOLUTION To assist with the solution, give the grid a chessboard colouring as shown.



The connections are on light and dark squares alternately; the path both starts and finishes on a light square; so there must be an odd number of connections overall. Each quarter–circle connection changes the direction of the path from horizontal to vertical, or *vice versa*; the path both starts and finishes vertically; therefore, there must be an even number of quarter–circles. Hence, there are an odd number of straight connections.