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Epsilon-Delta Definitions And Continuity

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Abstract

We present a formal epsilon-delta definition of a limit for real functions on the reals and on the real plane, and we give examples of how to derive and write up proofs that use this useful definition. A brief section on continuity with the epsilon-delta definition is also included.

1 Topology of the reals

Many fundamental ideas in calculus and analysis rely on the concept of two points being close to each other. We know how to measure the distance between two points in the plane and even in 3-dimensional Euclidian space. We further can generalize this distance formula to higher dimensions and calculate the distance between any n-dimensional points in Euclidian n-space, or $\mathbb{R}^n$. Now consider a function $f : \mathbb{R} \to \mathbb{R}$. What happens to the function values of $f(x)$ when x approaches some given value? Does $f(x)$ then also approach a constant value, or not? This is the notion of a limit, and it is purpose of this article to introduce a formal definition of this limit which is technical but useful and precise.

2 An epsilon-delta definition of limits for real functions

When examining the limit of a function f at some point c, we are trying to find the values of $f(x)$ as x gets infinitely close to c. To do this, the function f must be defined at points close to c, although not necessarily at c itself. But what is close? We are going to formulate an understanding of the limit by defining what close actually means.

Definition. *Let $f : U \to \mathbb{R}$ be a real function on a nonempty subset U of $\mathbb{R}$. Then $L \in \mathbb{R}$ is a* limit *of f at some* limit point *$c \in U$ if and only if, for each $\epsilon > 0$, there exists a value $\delta > 0$ such that*

$$|f(x) - L| < \epsilon \qquad \textit{whenever} \qquad x \in U \quad \textit{and} \quad 0 < |x - c| < \delta\,.$$

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Loosely speaking, this definition states that $f(x)$ has L as limit at point $x = c$ if $f(x)$ gets closer to L when x gets closer to c. In order to use this epsilon-delta definition for proving that a function has a limit at some given point, we will first work backwards to find some appropriate δ that will let us conclude from $|x - c| < \delta$ that $|f(x) - L| < \epsilon$. After finding this suitable delta value, we can then write the proof forwards.

Let us first consider a trivial example.

Example 1. Consider the constant function $f(x) = k$ for some $k \in \mathbb{R}$ and all $x \in U = \mathbb{R}$. Notice that $f(x)$ has the limit $L = k$ at any point $x = c \in \mathbb{R}$. This is obvious but an epsilon-delta proof of this observation could be as follows:

Proof. Let $c \in \mathbb{R}$ and $\epsilon > 0$, and set $\delta = \epsilon$. If $|x - c| < \delta$, then

$$|f(x) - f(c)| = |k - k| = 0 \leq |x - c| < \delta = \epsilon\,.$$

Hence, $f(x)$ has the limit $L = k$ at $c \in \mathbb{R}$. □

We begin now our less trivial examples of epsilon-delta proofs.

Example 2. Consider the function $f(x) = 5x - 3$. We are going to use an epsilon-delta proof to show that the limit of $f(x)$ at $c = 1$ is $L = 2$. In order to do that, we need to find, for each $\epsilon > 0$, a value $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \in U$ and $0 < |x - c| < \delta$. In other words, we want to find some $\delta > 0$ that will satisfy

$$|f(x) - L| = |(5x - 3) - 2| = |5x - 5| = 5|x - 1| < \epsilon$$

whenever $|x - c| = |x - 1| < \delta$. We could choose $\delta = \epsilon/5$ and can now write the proof.

Proof. Set $L = 2$, let $\epsilon > 0$ and choose $\delta = \epsilon/5$. If $|x - 1| < \delta$, then

$$|f(x) - L| = |(5x - 3) - 2| = 5|x - 1| < 5\delta = 5\epsilon/5 = \epsilon\,.$$

Hence, $f(x)$ has the limit $L = 2$ at $c = 1$. □

As we will soon see, the epsilon-delta proof with linear functions is very trivial compared even to quadratic functions. In the following example, the value $f(c)$ is not defined but, as we will see, this does not effect the proof.

Example 3. Consider the function $f : \mathbb{R} - \{1\} \to \mathbb{R}$ given by

$$f(x) = \frac{2x^2 - 3x + 1}{x - 1}$$

for all $x \in U = \mathbb{R} - \{1\}$, and note that

$$f(x) = \frac{(2x - 1)(x - 1)}{(x - 1)} = 2x - 1\,.$$

We will now prove that the limit of $f(x)$ at the point $c = 1$ is $L = 1$ - even though $f(1)$ is not defined. In order to do that, we need to find, for each $\epsilon > 0$, a value $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \in U$ and $0 < |x - c| < \delta$. In other words, we want to find some $\delta > 0$ that will satisfy

$$|f(x) - L| = |(2x - 1) - 1| = |2x - 2| = 2|x - 1| < \epsilon$$

whenever $|x - c| = |x - 1| < \delta$. We will choose $\delta = \epsilon/2$ and now complete the proof.

Proof. Set $L = 1$, let $\epsilon > 0$ and choose $\delta = \epsilon/2$. If $|x-1| < \delta$ and $x \neq 1$, then

$$|f(x) - L| = \left|\frac{2x^2 - 3x + 1}{x - 1} - 1\right| = 2|x-1| < 2\delta = 2\epsilon/2 = \epsilon\,.$$

Hence, $f(x)$ has the limit $L = 1$ at $c = 1$. □

Let us look into a more complicated example of a quadratic limit that will involve some more thought.

Example 4. Consider the function $f(x) = x^2 + x - 2$. Let us prove that the limit of this function as x approaches $c = 2$ is $L = 4$. We begin again by finding a suitable delta value so that

$$|f(x) - L| = |(x^2 + x - 2) - 4| = |x^2 + x - 6| = |(x+3)(x-2)| = |x+3||x-2| < \epsilon$$

whenever $|x - 2| < \delta$. Now, we cannot let delta be immediately expressed in terms of x and ϵ so we first have to simplify the problem. Since the concept of the limit only applies when x is close to $c = 2$, we may restriction the distance $|x - c|$ to less than 1, say, by setting δ to 1 or less. Expressed mathematically, this is $|x - c| = |x - 2| < \delta \leq 1$. Then $1 < x < 3$ so, in particular, $|x + 3| < 6$. In other to satisfy $|x+3||x-2| < \epsilon$, we can choose delta $\delta = \epsilon/6$. Thus, we are going to choose $\delta = \min\{1, \epsilon/6\}$ and can now write the proof.

Proof. Set $L = 4$, let $\epsilon > 0$, let $\delta = \min\{1, \epsilon/6\}$. If $|x - 2| < \delta$, then $|x - 2| < 1$ so $|x + 3| < 2 + 1 + 3 = 6$; hence,

$$\begin{aligned}|f(x) - L| &= |(x^2 + x - 2) - 4| \\ &= |(x+3)(x-2)| = |x+3||x-2| < 6\delta = 6\min\{1, \epsilon/6\} \leq 6\epsilon/6 = \epsilon\,.\end{aligned}$$

Hence, $f(x)$ has the limit $L = 4$ at $c = 2$. □

Although this epsilon-delta definition is helpful for most functions, there are cases where the limit does not exist, and this definition will prove that the limit is nonexistent. We see this in the following example.

Example 5. Consider the real function given by

$$f(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0\,. \end{cases}$$

Clearly, this function has no limit at $x = 0$. Let us now show this by using the epsilon-delta definition. Assume that $f(x)$ does indeed have some limit L at $c = 0$ and consider some positive $\epsilon < 1$. Then, since the two function values of f lie at distance 2 apart, it is not possible that $|f(x) - L| < 1$ for any x. In particular, there is no value of δ which this inequality would be true when $0 < |x - c| < \delta$. Therefore, $f(x)$ has no limit at $x = 0$ by the epsilon-delta definition of the limit.

3 The 2-variable case

We now shift our focus to the formal definition of the limit for real functions on the real plane. It is a straight-forward generalisation of our definition for real functions.

Definition. *Let $f : U \to \mathbb{R}$ be a real function on a nonempty subset U of $\mathbb{R}^2$. Then $L \in \mathbb{R}$ is a* limit *of f at some* limit point $\mathbf{c} \in U$ *if and only if, for each $\epsilon > 0$, there exists a value $\delta > 0$ such that*

$$|f(x) - L| < \epsilon \qquad \textit{whenever} \qquad \mathbf{x} \in U \quad \textit{and} \quad 0 < ||\mathbf{x} - \mathbf{c}|| < \delta\,.$$

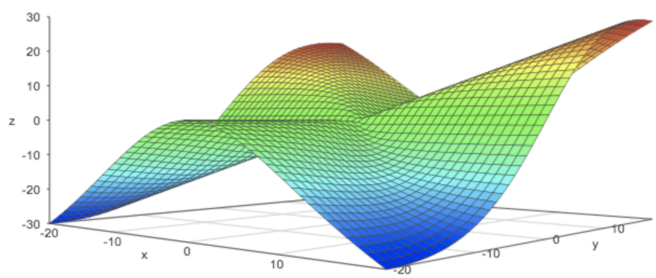
Here, the $||\mathbf{x}||$ notation denotes the Euclidean norm of each vector $\mathbf{x} = (x, y) \in \mathbb{R}$, namely

$$||\mathbf{x}|| = \sqrt{x^2 + y^2}\,.$$

Example 6. Let $f : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}$ be the function

$$f(x, y) = \frac{3x^2y}{x^2 + y^2}\,.$$

The following plot illustrates this function as a surface in $\mathbb{R}^3$ with points $\big(x, y, f(x, y)\big)$.



It appears that the limit of this function as x approaches $\mathbf{c} = (0, 0)$ is $L = 0$. Indeed, let us prove that that is the case, so consider some $\epsilon > 0$. Before reading the proof below, can you, dear Reader, find a suitable delta value so that, whenever $||\mathbf{x} - \mathbf{c}|| = ||\mathbf{x}|| < \delta$,

$$|f(x, y) - L| < \epsilon\,?$$

Proof. Set $L = 0$, let $\epsilon > 0$ and set $\delta = \epsilon/3$. If $||\mathbf{x} - \mathbf{c}|| < \delta$ for any $\mathbf{x} = (x, y) \in \mathbb{R}^2$, then

$$|f(x, y) - L| = \left|\frac{3x^2y}{x^2 + y^2} - 0\right| = \left|\frac{3x^2y}{x^2 + y^2}\right| = \frac{3x^2|y|}{x^2 + y^2} \le \frac{3(x^2 + y^2)|y|}{x^2 + y^2} = 3|y| = 3\sqrt{y^2}$$
$$\le 3\sqrt{x^2 + y^2} = 3||\mathbf{x}|| = 3||\mathbf{x} - \mathbf{c}|| < 3\delta = 3\epsilon/3 = \epsilon\,.$$

Thus, by the above definition, f has the limit is 0 at $\mathbf{c} = (0, 0)$. $\square$

A common approach to analyzing the limit of a multivariable function, like f above, is find the limit, if it exists, along any curve in the plane through the given limit point $\mathbf{c} \in U$, and to see whether such limits are the same for all curves. The epsilon-delta definition approach is at times easier, although the calculations can be complex. It gives you a solid answer in a precise and well organized manner.

It is straight-forward to generalise the epsilon-delta definition limit to real functions on $\mathbb{R}^n$. Instead, let us now end the paper with a discussion on continuity.

4 An epsilon-delta definition of continuity

We now move from analyzing the limit of a function at a point c to analyzing the continuity of a function f on $\mathbb{R}^2$. In the last section, we saw that the limit of a function $f(x)$ at a point c is independent of the value of the function $f(c)$. In the special case that the limit of f at c is equal to the value of $f(c)$, a function is said to be continuous at $x = c$. This section gives an epsilon-delta definition of continuity for functions on the real numbers. It is very similar to the limit definition, as follows.

Definition. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real function and let $c \in \mathbb{R}$. Then f is* continuous *at c if, for each $\epsilon > 0$, there exists a value $\delta > 0$ such that*

$$|f(x) - f(c)| < \epsilon \qquad \textit{whenever} \qquad x \in \mathbb{R} \quad \textit{and} \quad 0 < |x - c| < \delta\,.$$

Example 7. Consider the function $f(x) = 5x - 3$. We can show that this function is continuous at $c = 1$ by the above definition. In order to do that, we need to find, for each $\epsilon > 0$, a value $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $x \in \mathbb{R}$ and $0 < |x - c| < \delta$. In other words, we want to find some $\delta > 0$ that will satisfy

$$|f(x) - f(c)| = |(5x - 3) - (5 \times 1 - 3)| = |4x - 4| = 4|x - 1| < \epsilon$$

whenever $|x - c| = |x - 1| < \delta$. We could choose $\delta = \epsilon/4$ and can now proceed to the proof.

Proof. Set $c = 1$, let $\epsilon > 0$ and choose $\delta = \epsilon/4$. If $|x - 1| < \delta$, then

$$|f(x) - f(c)| = |(5x - 3) - (5 \times 1 - 3)| = |4x - 4| = 4|x - 1| < 4\delta = 4\epsilon/4 = \epsilon\,.$$

The function $f(x)$ is therefore continuous at $x = c = 1$. □

References

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