Parabola Volume 53, Issue 2 (2017)

On Prime Determinants

Michael Kielstra[1]

1 Introduction

It is obvious that all prime numbers greater than 2 are of the form $2n \pm 1$. It is well-known that all prime numbers greater than 2 are of the form $4n \pm 1$. It can be shown that all prime numbers greater than 3 are of the form $6n \pm 1$. When a number k has the property that all prime numbers greater than k are of the form $kn \pm 1$ where n is an integer greater than 0, we say that k is a *prime determinant*. (I am unaware of any term commonly in use for such numbers and so have invented my own; should any readers know of such a term, please contact me.) In this paper, I will prove that 1, 2, 3, 4, 6 are prime determinants, and give reasons why no other numbers are.

2 Prime determinants

Theorem 1. 1, 2, 3, 4, 6 *are each prime determinants.*

Proof. It is obvious that any prime number can be written as $1n \pm 1$; similarly, it is obvious that any prime greater than 2 can be written as $2n \pm 1$.

All prime numbers greater than 3 can be written as $3n + 1$ or $3n + 2 = 3(n + 1) - 1$, so any prime number greater than 3 can be written as $3n + 1$ or $3n - 1$.

Neither $4n$ nor $4n + 2$ can be prime, so any prime greater than 4 can be written as $4n+1$ or $4n+3 = 4(n+1)-1$. Hence, any prime greater than 4 can be written as $4n\pm1$.

Finally, since none of $6n$, $6n + 2$, $6n + 3$, and $6n + 4$ can be prime, any prime number greater than 6 must be of the form $6n + 1$ or $6n + 5 = 6(n + 1) - 1$. Again, this implies that each such prime can be written as $6n \pm 1$. □

3 Factorisation of prime determinants

To characterise the set of prime determinants, we begin by considering the factorisation of any given member. Note here that it is not necessary, for p to be a prime determinant, that all integers $pn \pm 1$ are prime. Indeed, that would be impossible (see the Appendix).

Theorem 2. *No prime determinant* p *is coprime to an integer between* 1 *and* $p - 1$.

[1]Michael Kielstra is a high school student living in English and is author of the weekly mathematics blog wanderingmathematician.wordpress.com.

Proof. Let us assume, for a proof by contradiction, that there is some integer q between 1 and $p-1$ that is coprime to p. Note that no number of the form $np+q$ can be written in the form $mp \pm 1$, as that would imply $(m-n)p = q \pm 1$ and therefore that $p \mid q \pm 1$. Since $q < p-1$, $p \nmid q \pm 1$. (This has no bearing on whether or not q divides p).

Therefore, if p is coprime to q and p is also a prime determinant, then there must be no primes of the form $np+q$. Otherwise, there would be a prime greater than p that could not be written in the form $np \pm 1$. To rephrase, the arithmetic progression with first term q and common difference p must contain no primes. Sadly, as shown in an 1837 paper by Dirichlet, any arithmetic progression where the first term and the difference are coprime will contain infinitely many primes [1]. Therefore, no prime determinant may be coprime to any number between 1 and one less than itself. □

Theorem 3. *Any integer p that is not coprime to any other integer between 1 and $p-1$ is a prime determinant.*

Proof. Note that each integer x can be written in the form $np+q$, where $q < p$, for any given p. If x is prime, then the greatest common denominator d of p and q must equal 1 because, otherwise, we could factor out d to show that $np+q$ is composite. However, $d > 1$ for all $1 < q < p-1$. This implies that, for $np+q$ to be prime, either $q = 1$ or $q = p-1$. Both of these, however, reduce to $x = np \pm 1$. In short, all integers can be written as $np+q$ but $np+q$ is only prime when $q = 1$ or $p-1$. □

Combining these two theorems, we may now state the following:

Theorem 4. *An integer p is a prime determinant if and only if it is not coprime to any integer q such that $1 < q < p-1$.*

4 A very exciting table

We have established a simple algorithmic test to determine whether or not any given integer is a prime determinant. It would now be useful to examine a table of the first few integers, together with the reasons why they are or are not prime determinants.

Number	Prime determinant?	Reason
1	Yes	No integers between 1 and $1-1=0$
2	Yes	No integers between 1 and $2-1=1$
3	Yes	No integers between 1 and $3-1=2$
4	Yes	Not coprime to 2
5	No	Coprime to $2 \in \{2,3\}$
6	Yes	Not coprime to $2,3,4$
7	No	Coprime to $2 \in \{2,\ldots,5\}$
8	No	Coprime to $3 \in \{2,\ldots,6\}$
9	No	Coprime to $2 \in \{2,\ldots,7\}$
10	No	Coprime to $3 \in \{2,\ldots,8\}$.

5 Upper bounds on prime determinants

The table seems to imply the following theorem:

Theorem 5. *There are no prime determinants greater than* 6*.*

Before proving this, we state the following lemma.

Lemma 6. *No integer* p *has a factor between* $\frac{p}{2}$ *and* p*.*

Proof. Assume that q is a factor of p with $\frac{p}{2} < q < p$. Then there is some r such that $qr = p$; hence, $qr < 2q < 2qr$. But then $r < 2 < 2r$. This is a contradiction – there is clearly no r less than 2 such that $2 < 2r$. Therefore, p has no factor between $\frac{p}{2}$ and p. □

We are now in a position to prove Theorem 5. According to Bertrand's Postulate, otherwise known as the Bertrand-Chebyshev Theorem or Chebyshev's Theorem, for all integers $p > 3$, there is a prime number between p and $2p - 2$ [2]. (During my researches, I came across an easy way to remember this postulate: "Chebyshev said it, I'll say it again: there's always a prime between n and $2n$.")

If there is a prime number less than a prime determinant, then the factorisation of the determinant must include that prime number. Otherwise, they would be coprime to each other, and Theorem 4 would be false.

However, for all integers $p > 6$, there is a prime between $\frac{p}{2}$ and $p - 1$. Therefore, no integer greater than 6 can be a prime determinant. As in the table, 5 is also not a prime determinant, so the only integers that remain are 1, 2, 3, 4, and 6. Thus we state the following, climactic, theorem:

Theorem 7. *The only prime determinants are* 1*,* 2*,* 3*,* 4*, and* 6*.*

6 Conclusion

Prime numbers are famously both ordered and chaotic at the same time. The graph of the function $\pi(n)$, the number of primes up to n, is jagged and seemingly random. However, primes spring from the inherently deterministic process of factorising, and thus must have some kind of pattern. The interested reader should look up the Riemann Hypothesis for a conjecture, as yet unproven, that promises to deliver the ultimate secret of the primes and produce a formula for $\pi(n)$. Until then, mathematicians must content themselves with smaller "factoids" such as the ones proven here.

References

[1] Peter Gustav Lejeune Dirichlet. There are infinitely many prime numbers in all arithmetic progressions with first term and difference coprime. 1837.

[2] Jonathan Sondow and Eric W. Weisstein. Bertrand's postulate. http://mathworld.wolfram.com/BertrandsPostulate.html, Retrieved on 10 October 2017.