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Unit Length Wheel Graph Embedded in $\mathbb{R}^n$

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1 Introduction

I have always loved math for as long as I can remember. Ever since I was little, I wanted to one day contribute something new to mathematics. That day finally came; one day during the last summer, I was sitting and relaxing at a coffee shop. I saw a small spinning water wheel model. For no particular reason, I started to google wheel related contents. After reading some random pages across the Internet, I came across the article *Wheel graph* on Wikipedia. My first impression was, "Wow, everything can be related to math!" I was curious enough to read more about it later on and eventually came up with some original questions. I then started my research on it with a friend. Our research was fruitful after some effort.

When I first asked my friend to collaborate on this research, she asked me: "What is it good for?" This question took me by surprise. I never thought about the application of this. Well, I thought about it and told her: "Because it is beautiful and fun." Mathematical knowledge is beautiful in and of itself. Sometimes, the only way to appreciate its beauty is to let your imagination fly (for example, there is no way to depict a true four dimensional space since we only perceive three spatial dimensions). The Greek mathematicians were pioneers in pure mathematics who found the knowledge worthy on its own. They were among the first who truly appreciated the absolute value of mathematics. There is probably infinite knowledge out there. We are merely explorers of infinity, and there is much more waiting for a daring explorer to discover. While it is true that this research is in no way revolutionary, its beauty remains attractive to me. In this paper, we are going to explore how many dimensions it takes to embed a wheel graph with multiple hubs in Euclidean space.

Mathematics is everywhere among us. One of *Parabola*'s more recent papers might have been inspired by ordinary objects such as the street tiling found in Cairo. In my case, it was the water wheel. Hence, next time you want aspiration for a mathematics problem, just look around your house; there may be an inspirational object laying around somewhere.

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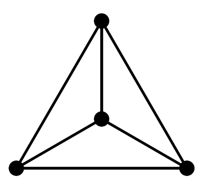
2 History

About half a century ago, Paul Erdös and his research associates [?] invented the idea of a wheel graph and proved some results on these graphs. Erdös was one of the most prolific mathematicians of the 20th century. In popular mathematics culture, he was perhaps most known through the *Erdös number* which was created due to his massive number of collaborations around the world. The Erdös number is the collaborative distance from Erdös. An Erdös number of 1 means that person has co-authored with Erdös, and a number of 2 is a person who has co-authored with someone with an Erdös number of 1. Fun fact: a famous 18th century mathematician, Pierre-Simon Laplace, has an Erdös number of 14 [9].

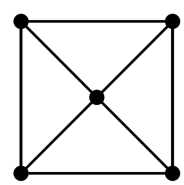
About two decades later, two mathematicians, Fred Buckley and Frank Harary [?], proved results on the embeddings of wheel graph. Three decades later, my friend and I happened to run across this problem by ourselves and proved similar results, presented here in this paper. Little did we know that our problem can be traced back over half a century ago. However, Parabola's editor Thomas Britz pointed this out, and it was such an honor for us to work on a problem that was introduced by someone like Erdös. This opens our eyes for some bigger meaning that mathematics is a continuous flow that connects the past to the present and beyond, and we are glad to take a part in this flow.

3 Non-crossing Euclidean embeddings of wheel graphs

The *wheel graph* W_n consists of n dots, called *vertices*, $n-1$ of which are connected circularly by lines, called *edges*, and which are also connected by edges to a central vertex, called the *hub*. The non-hub vertices are called *rim vertices*. By allowing k hubs, we can define the more general wheel graphs W_n^k. Here are two small wheel graphs.



W_4



W_5

In this paper, we are trying to answer the question:

What is the smallest dimension m for which the wheel graph W_n^k can be drawn in $\mathbb{R}^m$ so that all edges have length 1 and are not allowed to cross?

We will call such a drawing of W_n^k a *non-crossing Euclidean embedding* in $\mathbb{R}^m$. This smallest dimension m is called the *non-crossing Euclidean dimension* and is denoted by $e(W_n^k)$. Euclidean embeddings of wheel graphs and other graphs, where crossing were allowed, were studied by Paul Erdős, Frank Harary, William Tutte and others; see [1–8]. Our results are related to their results but are different due to the restriction of edges not being allowed to cross.

4 One hub

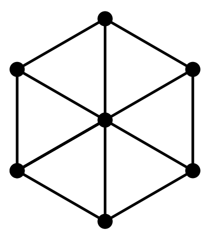
We first consider wheel graphs with just one hub.

Theorem 1. $e(W_n) = \begin{cases} 2 & , n = 7 \\ 3 & , n \neq 7 \end{cases}$

Proof. For W_4, draw the three rim vertices as the points of an equilateral triangle in the xy plane of $\mathbb{R}^3$, centered in $(0, 0, 0)$. We now draw the hub on the z-axis, at $(0, 0, z)$. It is easy to see that z can be found so that all edges of W_4 have unit length. Thus, $e(W_4) = 3$.

We can use this construction to draw any wheel W_n in $\mathbb{R}^3$. For suppose that we have a non-crossing Euclidean embedding of W_n. Then consider any two rim vertices u and v that are connected by an edge. Together with the hub, these vertices and their connecting edges form an equilateral triangle. By our construction, we can find another vertex w so that all four vertices are at unit distance to each other. Now draw an edge from w to the hub and to u and v, and delete the edge between u and v. This gives a non-crossing Euclidean embedding of W_{n+1} in $\mathbb{R}^3$, so, by induction, $e(W_n) \leq 3$ for all n.

The wheel graph W_7 is the only wheel graph that has a unit embedding in $\mathbb{R}^2$:

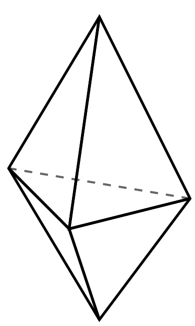


W_7

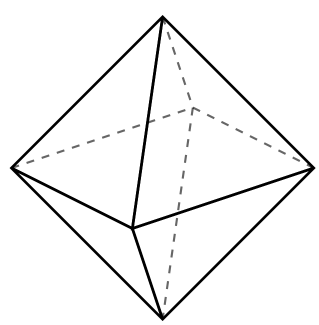
□

5 Two hubs

We now consider non-crossing Euclidean embeddings of wheel graphs of two hubs, such as the following:



W_5^2



W_6^2

Theorem 2. $e(W_n^2) = 3$ *for* $n = 5, 6, 7$ *and* $e(W_n^2) \geq 4$ *for* $n \geq 8$.

Proof. If W_n^2 has a non-crossing embedding in $\mathbb{R}^m$, then so has W_{n-1}; just delete one of the two hubs and its incident edges from W_n^2 to get W_{n-1} (both have $n-2$ rim vertices). Therefore, $e(W_n^2) \geq e(W_{n-1}) = 3$ for $n \neq 8$. For $n = 5, 6, 7$, the constructions for W_{n-1} in the proof of Theorem 1 can be used to construct non-crossing Euclidean embeddings in $\mathbb{R}^3$ for W_n^2. In particular, the rim vertices all lay in a plane so add the hub vertex and its incident edges are are mirror images in this plane of the first hub and its inciddents edges. This shows that $e(W_n^2) \leq 3$ for $n = 5, 6, 7$, so $e(W_n^2) = 3$ for these values.

For $n \geq 8$, assume that $e(W_n^2) = 3$. Then $e(W_n^2)$ has a non-crossing Euclidean embedding in $\mathbb{R}^3$. The rim vertices lie at distance 1 from each hub and therefore lie in the intersection of the two unit spheres around the two hubs; that intersection is a lower-dimensional sphere, namely a circle. If no edges may cross, then the rim vertices must lie evenly on a circle with the edges forming a convex $(n-2)$-gon. However, the distance from the rim vertices to each hub is then 1 when $n = 8$ or more when $n \geq 9$, so there cannot be two hubs when $n = 8$ or indeed any hub when $n \geq 9$. This a contradiction, so $e(W_n^2) \geq 4$. □

6 Arbitrarily many hubs

We now consider non-crossing embeddings of W_n^k with $n \geq 3$ hubs. Ignoring the trivial cases in which there are at most two rim vertices, we have the following theorem.

Theorem 3. *For* $k \geq 3$, $e(W_n^k) = \begin{cases} 4 & , \ n-k \in \{3,4,5\} \\ 5 & , \ n-k \geq 6 \, . \end{cases}$

Proof. Suppose that there are $n - k \geq 3$ rim vertices and assume that there is a non-crossing embedding of W_n^k in $\mathbb{R}^3$. Then the rim vertices lie at equal distance to any two hubs and thus lie in a plane. This is true of each three of the pairs of hubs considered, so the rim vertices line in the intersection of three planes, which is either a point or a line. But this cannot be since the rim vertices cannot then have unit distances between any two of them. Therefore, $e(W_n^k) \geq 4$.

Now consider the case in which there are exactly $n-k = 3$ rim vertices, respectively. Define the three rim vertices to lie at points of the form $(x, y, 0, 0)$ where $x^2 + y^2 = \frac{1}{3}$; these form an equilateral triangle with side lengths 1. We will now choose three hub vertices with coordinates of the form $(0, 0, z_1, z_2)$. These must necessarily satisfy the equation

$$x^2 + y^2 + z_1^2 + z_2^2 = 1$$

where x and y are the first two coordinates of any rim vertex. Since $x^2 + y^2 = \frac{1}{3}$, the hub vertices must satisfy the equation

$$z_1^2 + z_2^2 = \frac{2}{3}\,;$$

choose a set of k solutions for the k hub coordinates. Drawing lines between rims and hubs provides us with non-crossing embeddings of W_n^k in $\mathbb{R}^4$, so $e(W_n^k) = 4$.

The cases in which there are $n-k=4$ or $n-k=5$ rim vertices are similar: instead of an equilateral triangle, draw a square and a pentagon, respectively.

Now consider the case in which we have $n-k \geq 6$ rim vertices. Use the previously described inductive construction to obtain a non-crossing Euclidean embedding of W_{n-k+1} in $\mathbb{R}^3$. By adding two zero coordinates to each of the vertices (w,x,y) in W_{n-k+1}, we now have one hub and $n-k$ rim vertices drawn as points $(w,x,y,0,0)$ in $\mathbb{R}^5$. Now, in the same was as above, find hub vertices of the form $(0,0,0,z_1,z_2)$. Drawing lines between rims and hubs provides us with non-crossing embeddings of W_n^k in $\mathbb{R}^5$, so $e(W_n^k)=5$. $\square$

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