Parabola Volume 53, Issue 2 (2017)

Problems 1531–1540

Parabola would like to thank Peter Brown[1] for writing the problems for this issue, and to Akash Pardeshi[2] for contributing Problem 1540.

Q1531 Take any four consecutive whole numbers, multiply them together and add 1. Make a conjecture and prove it!

Q1532 Let ABC be a triangle with longest side BC and let P be a point in the interior of the triangle. Show that $AP < BP + PC$.

Q1533

(a) Show that if p is odd, then $x^p - 1 = (x-1)(x^{p-1} + x^{p-2} + \cdots + x + 1)$.

(b) Hence, show that if x is odd, then the highest power of 2 which divides $x^p - 1$ also divides $x - 1$.

(c) Find the highest power of 2 which divides $1999^{2000} - 1$.

Q1534 Let $ABCD$ be a convex quadrilateral and let P, Q, R, S be points on AB, BC, CD, DA respectively, such that $AP = \frac{1}{4}AB$, $QC = \frac{1}{4}BC$, $CR = \frac{1}{4}CD$ and $SA = \frac{1}{4}DA$.

(a) Show that $PQRS$ is a parallelogram.

(b) Find the ratio of the area of $PQRS$ to that of $ABCD$.

Q1535 Find all positive integer solutions to $2x^2 - 2xy + y^2 = 65$.

Q1536 The base notation a_b appearing in this problem is mostly recently explained in the *Parabola* article here.

(a) Show that whatever base b is used, the number 21_b is never equal to twice 12_b.

(b) Find all the numbers and all bases $b \leq 12$ for which there exists a two digit number ac_b which is twice the number obtained by reversing its digits.

(c) Find all bases b and all numbers $n = ac_b$ such that $n = 2 \times ac_b$.

Q1537 Denote the top of a cube by $ABCD$ and the bottom by A_1, B_1, C_1, D_1, so that A is directly above $A_!$ and so on. Take midpoints of the six edges AB, BB_1, B_1C_1, C_1D_1, D_1D and DA. Show that a plane containing any three of these points contains them all and deduce that these points form the vertices of a regular hexagon.

---

[1]Peter Brown is a Senior Lecturer at UNSW Sydney.
[2]Akash Pardeshi is a rising sophomore at the University of Illinois Laboratory High School.

Q1538

(a) Simplify $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$.

(b) Show that $(a^2+b^2+c^2-ab-ac-bc) \geq 0$.

(c) Prove that if x, y, z are positive real numbers, then $\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$.

Q1539 If we expand $(2+x)^{18}$ as a polynomial, then we obtain

$$(2+x)^{18} = a_0 + a_1x + a_2x^2 + \cdots + a_{18}x^{18}\,,$$

where $a_0, a_1, \ldots, a_{18}$ are integers.
Without using the Binomial Theorem), find a_0, a_1, a_{18} and $a_0 + a_1 + \cdots + a_{18}$.

Q1540 Circumscribe a right circular cone about a sphere of radius r such that the cone has minimum volume. Prove that the cone has exactly double the volume of the sphere.