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Solutions 1521–1530

Q1521 Solve the equation

$$\sqrt{x+20}+\sqrt{x}=17\,. \tag{$*$}$$

SOLUTION There are many ways to solve the equation but perhaps this is the simplest: using a difference of two squares, we have

$$\big(\sqrt{x+20}+\sqrt{x}\big)\big(\sqrt{x+20}-\sqrt{x}\big)=(x+20)-x=20\,.$$

By dividing this by equation $(*)$, we get

$$\sqrt{x+20}-\sqrt{x}=\frac{20}{17}\,,$$

and now subtracting this from $(*)$ gives

$$2\sqrt{x}=17-\frac{20}{17}\,.$$

Therefore,

$$x=\Big(\frac{1}{2}\Big(17-\frac{20}{17}\Big)\Big)^2=\Big(\frac{269}{34}\Big)^2\,.$$

Q1522 Consider a circle with centre $O=(a,b)$ and radius r, and a point $P=(p,q)$ which lies outside the circle. If Q and R are the two points on the circle such that PQ and PR are tangent to the circle, find the equation of the line QR.

SOLUTION Let $T=(x,y)$ be any point on the circle so that OT is perpendicular to PT. By the "gradients of perpendicular lines" theorem, we have

$$\frac{y-b}{x-a}\,\frac{y-q}{x-p}=-1\,,$$

which can be rewritten as

$$(x-a)(x-p)+(y-b)(y-q)=0\,;$$

and, since T is on the circle, we have

$$(x-a)^2+(y-b)^2=r^2\,.$$

Now subtract the previous equation from this one to get

$$(p-a)(x-a)+(q-b)(y-b)=r^2\,. \tag{$*$}$$

Now note that (i) this is the equation of a line since it has the form $Ax+By=C$ for certain constants A,B,C with A,B not both zero; and (ii) both P and Q lie on this line since both $T = P$ and $T = Q$ satisfy the conditions OT on the circle and $OT \perp PT$ ("tangent perpendicular to radius" theorem). So $(*)$ is our answer!

Comment. The "gradients of perpendicular lines" equation does not make sense if OT is horizontal and PT is vertical, or *vice versa*; but the following equation is still correct, as you may easily check, and so our solution is still correct.

Q1523 Consider a list of the first n positive integers in some order; for example, if $n = 7$, then we could have $5, 1, 6, 4, 7, 3, 2$. We seek lists which have the following further property: for every integer k from 1 to n, the sum of the first k numbers in the list is a multiple of k. For instance, the above list **does not** have this property since the sum of the first 5 numbers is $5 + 1 + 6 + 4 + 7 = 23$ which is not a multiple of 5.

Find all lists with this property.

SOLUTION We shall use the notation $a \mid b$ to indicate that a is a factor of b (that is, b is a multiple of a).

First, we show that if we have a list of n numbers with the stated property, then n cannot be even. Suppose that $n = 2m$. Applying the property to $k = 2m$, we have

$$2m \mid a_1 + a_2 + \cdots + a_{2m} \,.$$

But the numbers $a_1, a_2, \ldots, a_{2m}$ are just $1, 2, \ldots, 2m$. Their order might possibly be different but this does not affect their sum. So

$$2m \mid 1 + 2 + \cdots + 2m = m(2m + 1)$$

and so $2 \mid 2m + 1$, which is impossible. Therefore, n cannot be even...

...and so n is odd, say $n = 2m + 1$. We shall show that this is also impossible if $m > 1$. We have

$$n - 1 \mid a_1 + a_2 + \cdots + a_{n-1} = (a_1 + a_2 + \cdots + a_n) - a_n$$

so

$$2m \mid (2m + 1)(m + 1) - a_n = 2m^2 + 3m + 1 - a_n \,,$$

and $2m \mid m + 1 - a_n$. But a_n is a number from 1 to $2m + 1$, and so

$$-2m < -m \le m + 1 - a_n \le m < 2m \,;$$

if $m + 1 - a_n$ is a multiple of $2m$, then it must be zero, and so $a_n = m + 1$. Now, consider the same kind of thing at the previous step:

$$n - 2 \mid a_1 + a_2 + \cdots + a_{n-2} - (a_1 + a_2 + \cdots + a_n) - a_n - a_{n-1} \,,$$

so

$$2m - 1 \mid (2m + 1)(m + 1) - (m + 1) - a_{n-1} = (2m - 1)(m + 1) + m + 1 - a_{n-1} \,,$$

and $2m-1 \mid m+1-a_{n-1}$. For the same reasons as above, we have

$$-(2m-1) < -m \leq m+1-a_n \leq m < 2m-1\ ,$$

noting that $2m-1 > m$ since $m > 1$; thus, $a_{n-1} = m+1$, which is impossible since a_{n-1} and a_n are supposed to be *different* numbers.

The only remaining cases are $n = 1$ and $n = 3$, and it is easy to find by trial and error all possible lists

$$1 \quad \text{and} \quad 1,3,2 \quad \text{and} \quad 3,1,2\,.$$

Q1524 Find all positive integers n such that

$$1122^{n-1} + 2244^{n-1}$$

is a factor of

$$1122^n + 2244^n\,.$$

SOLUTION It is obvious that $n = 1$ is a solution; now we look for solutions $n \geq 2$. We seek integers n and k such that

$$k(1122^{n-1} + 2244^{n-1}) = 1122^n + 2244^n\,.$$

This can be written as $k(1 + 2^{n-1}) = 1122(1 + 2^n)$, which leads to

$$(2244-k)(1 + 2^{n-1}) = 1122\,.$$

Therefore, $1 + 2^{n-1}$ is an odd factor of 1122; since $1122 = 2 \times 3 \times 11 \times 17$, its odd factors are

$$1\,,\ 3\,,\ 11\,,\ 17\,,\ 33\,,\ 51\,,\ 187\,,\ 561\,;$$

however, not all of these can be equal to $1 + 2^{n-1}$, and we get only

$$1 + 2^1 = 3\,, \quad 1 + 2^4 = 17\,, \quad 1 + 2^5 = 33\,.$$

So, $1122^{n-1} + 2244^{n-1}$ is a factor of $1122^n + 2244^n$ for $n = 1, 2, 5, 6$.

Q1525 If the expression

$$(1-x)(1+2x)(1-3x)(1+4x)\cdots(1-(2n-1)x)(1+2nx)$$

is expanded and its terms collected, then what is the coefficient of x^2?

SOLUTION First multiply the terms together in pairs to get

$$(1+x-2x^2)(1+x-12x^2)\cdots(1+x-(2n-1)(2n)x^2)\,. \qquad (*)$$

Now there are two ways to get x^2 terms by multiplying out these quadratics:

- multiply x terms from two factors by 1 from every other factor;

- multiply an x^2 term from one factor by 1 from every other factor.

In the first case, the coefficient of each x^2 expression will be 1; and the number of ways in which we can get an x^2 term in this way is the number of ways that we can choose two factors from the n factors in $(*)$; that is, $C(n,2)$. Therefore, this is the coefficient of x^2 in the first case. In the second case, the coefficient of each x^2 expression will be the same as it is in $(*)$, and the sum of all these coefficients will be

$$-(1\times 2)-(3\times 4)-\cdots-(2n-1)(2n)\,.$$

To add this up, we shall use the formula for the sum of an arithmetic progression, which you should know, and another useful formula which you may not know:[1]

$$1^2+2^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)\,.$$

The coefficient that we need can be written

$$\begin{aligned}&-(2^2-2)-(4^2-4)-\cdots-((2n)^2-(2n))\\&\qquad=-(2^2+4^2+\cdots+(2n)^2)+(2+4+\cdots+2n)\\&\qquad=-4(1^2+2^2+\cdots+n^2)+2(1+2+\cdots+n)\\&\qquad=-4\,\frac{n(n+1)(2n+1)}{6}+2\,\frac{n(n+1)}{2}\;;\end{aligned}$$

and so the total coefficient of x^2 is

$$\begin{aligned}&-4\,\frac{n(n+1)(2n+1)}{6}+2\,\frac{n(n+1)}{2}+C(n,2)\\&\qquad=-4\,\frac{n(n+1)(2n+1)}{6}+2\,\frac{n(n+1)}{2}+\frac{n(n-1)}{2}\\&\qquad=-\frac{n(8n^2+3n+1)}{6}\,.\end{aligned}$$

Q1526 Some questions about continued fractions – for basic information on these, see Peter Brown's article in the previous issue.

(a) Find the continued fraction for $\sqrt{23}$.
(b) Find the continued fraction for $\frac{1}{3}\sqrt{11}$.
(c) Find the value of the continued fraction $[1;\,2,\,\overline{3,\,4}]$.

SOLUTION

(a) We write $\sqrt{23}$ as an integer plus a remainder; do the same with the reciprocal of the remainder; and keep going until the calculations repeat:

$$\sqrt{23}=4+(\sqrt{23}-4)$$

[1] See also the *Parabola* article "Proof by picture: A selection of nice picture proofs".

$$\begin{aligned}
\frac{1}{\sqrt{23}-4} &= \frac{\sqrt{23}+4}{7} = 1+\frac{\sqrt{23}-3}{7}\\
\frac{7}{\sqrt{23}-3} &= \frac{\sqrt{23}+3}{2} = 3+\frac{\sqrt{23}-3}{2}\\
\frac{2}{\sqrt{23}-3} &= \frac{\sqrt{23}+3}{7} = 1+\frac{\sqrt{23}-4}{7}\\
\frac{7}{\sqrt{23}-4} &= \sqrt{23}+4 = 8+(\sqrt{23}-4)
\end{aligned}$$

Since the last remainder is one that we have seen before in the first step, the entire calculation will repeat starting at the second step. Therefore,

$$\sqrt{23} = [4;\ \overline{1,\ 3,\ 1,\ 8}]\,.$$

Note that this expression has the "special form" referred to in Peter's article for the continued fraction of $\sqrt{N}$: the last number in the recurring section is twice the initial number, and the previous part $1,\ 3,\ 1$ of the recurring section is palindromic (the same backwards and forwards).

(b) Using the same procedure,

$$\begin{aligned}
\frac{\sqrt{11}}{3} &= 1+\frac{\sqrt{11}-3}{3}\\
\frac{3}{\sqrt{11}-3} &= \frac{3\sqrt{11}+9}{2} = 9+\frac{3\sqrt{11}-9}{2}\\
\frac{2}{3\sqrt{11}-9} &= \frac{\sqrt{11}+3}{3} = 2+\frac{\sqrt{11}-3}{3}
\end{aligned}$$

and so

$$\frac{\sqrt{11}}{3} = [1;\ \overline{9,\ 2}]\,.$$

(c) Write

$$\beta = [3;\ 4,\ \overline{3,\ 4}] = 3+\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{4+\cdots}}}\,.$$

Then

$$\beta = 3+\cfrac{1}{4+\cfrac{1}{\beta}}$$

from which we can evaluate β; and the number we want is

$$\alpha = 1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cdots}}} = 1+\cfrac{1}{2+\cfrac{1}{\beta}}\,.$$

Calculations yield

$$\beta = 3 + \frac{\beta}{4\beta + 1} = \frac{13\beta + 3}{4\beta + 1}$$

so

$$4\beta^2 + \beta = 13\beta + 3\ ,$$

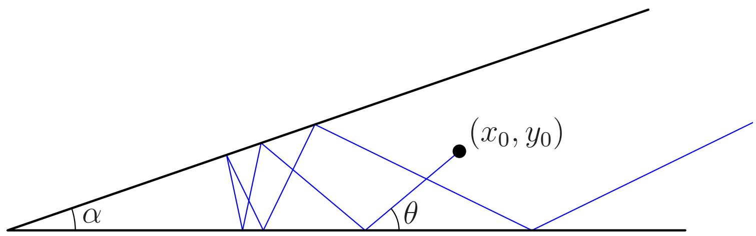
and solving the quadratic equation gives the positive root

$$\beta = \frac{3 + 2\sqrt{3}}{2}\ .$$

Hence,

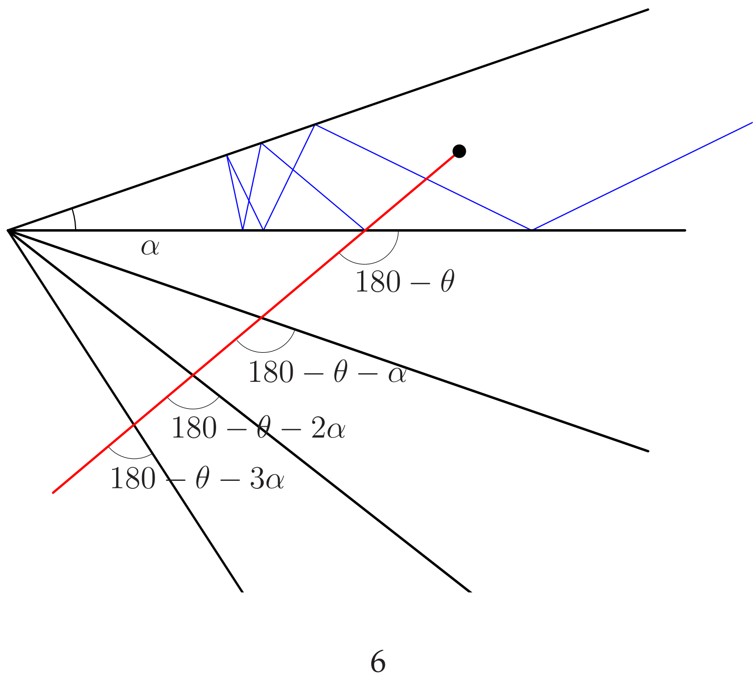
$$\alpha = 1 + \frac{\beta}{2\beta + 1} = \frac{3\beta + 1}{2\beta + 1} = \frac{4 + \sqrt{3}}{4}\ .$$

Q1527 In Problem 1517 we considered a billiard ball being projected into a "wedge–shaped table" as shown in the diagram.



If the angle between the ball's initial trajectory and the horizontal is θ, and the angle at the vertex of the wedge is α, how many times does the ball hit the wedge?

SOLUTION As in the solution to Problem 1517, we imagine that instead of the wedge remaining fixed and the ball being reflected, the wedge is reflected and the ball keeps going in a straight line. The successive angles made by the ball as it "exits" a wedge are $180 - \theta$, $180 - \theta - \alpha$, $180 - \theta - 2\alpha$ and so on, as shown in the diagram.



Now, the "exit angle" after the ball hits the wedge for the kth time is $180 - \theta - (k-1)\alpha$; and the ball will never hit the wedge again if this is less than or equal to the angle α of the wedge. So, the number of hits is the smallest k such that

$$180 - \theta - (k-1)\alpha \le \alpha \;;$$

that is, the smallest k such that

$$k \ge \frac{180-\theta}{\alpha} \;;$$

that is, $(180-\theta)/\alpha$, rounded to the nearest integer upwards.

Q1528 The first question should be easy; the second harder; the third harder still!

(a) Find the coordinates of three points P_1, P_2, P_3 such that the distance between any two of them is one unit. If the point M is equidistant from P_1, P_2, P_3, then find the distance MP_1.

(b) The same, but for four points.

(c) The same, but for five points.

SOLUTION It is clear that we can take our three points to be the vertices of an equilateral triangle. For example, let $P_1 = (0,0)$ and $P_2 = (1,0)$ and $P_3 = (a,b)$. Then, it is easy to see that we need

$$a = \frac{1}{2} \qquad \text{and} \qquad a^2 + b^2 = 1 \,,$$

so $b^2 = \frac{3}{4}$, and so $b = \frac{\sqrt{3}}{2}$, where we have chosen a positive value of b since only one solution was asked for. Hence, $P_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Write $M = (x,y)$; as M is equidistant from P_1 and P_2, we have $x = a = \frac{1}{2}$, and then $MP_1^2 = MP_3^2$. In other words,

$$x^2 + y^2 = (y-b)^2 \,,$$

so $\frac{1}{4} = -\sqrt{3}\,y + \frac{3}{4}$, and $y = \frac{1}{2\sqrt{3}}$. Therefore, $M = (\frac{1}{2}, \frac{1}{2\sqrt{3}})$. The distance from M to each point is

$$MP_1 = \sqrt{x^2+y^2} = \tfrac{1}{\sqrt{3}} \,.$$

We can check this trigonometrically by noting that the angle $\angle MP_1P_2$ will be $\frac{\pi}{6}$, so $y = \frac{1}{2}\tan\frac{\pi}{6}$ and $MP_1 = \frac{1}{2}\sec\frac{\pi}{6}$.

To find four points with unit distance between any two, we must begin by taking three of them to form an equilateral triangle as above, and then there is no way to obtain a fourth point equidistant from all three…

…unless we go into a third dimension. So, let $P_1 = (0,0,0)$ and $P_2 = (1,0,0)$ and $P_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ and $P_4 = (a,b,c)$. Then P_4 must be directly above (or below if you like) the point M from the previous question; so,

$$a = \frac{1}{2} \qquad \text{and} \qquad b = \frac{1}{2\sqrt{3}} \qquad \text{and} \qquad a^2 + b^2 + c^2 = 1 \,.$$

Hence, $c^2 = \frac{2}{3}$, and $c = \frac{\sqrt{2}}{\sqrt{3}}$; thus, $P_4 = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}})$. Let $M = (x, y, z)$; then $x = a = \frac{1}{2}$ and $y = b = \frac{1}{2\sqrt{3}}$ and $MP_1^2 = MP_4^2$. Hence,

$$x^2 + y^2 + z^2 = (z - c)^2 \,,$$

so

$$z = \frac{c^2 - x^2 - y^2}{2c} = \tfrac{1}{2\sqrt{6}} \,,$$

giving $M = (\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}})$ and $MP_1 = \sqrt{x^2 + y^2 + z^2} = \frac{3}{2\sqrt{6}}$. To find five points all unit distance apart...

...we need a fourth dimension! Let

$$\begin{aligned} P_1 &= (0,0,0,0) \\ P_2 &= (1,0,0,0) \\ P_3 &= (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0, 0) \\ P_4 &= (\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}, \tfrac{1}{2\sqrt{6}}, 0) \\ P_5 &= (a, b, c, d) \,. \end{aligned}$$

As in previous parts, we shall need

$$a = \frac{1}{2} \qquad \text{and} \qquad b = \frac{1}{2\sqrt{3}} \qquad \text{and} \qquad c = \frac{1}{2\sqrt{6}} \qquad \text{and} \qquad a^2 + b^2 + c^2 + d^2 = 1 \,.$$

Hence, $d^2 = \frac{5}{8}$, and $d = \frac{\sqrt{5}}{2\sqrt{2}}$. Then, $M = (x, y, z, w)$ with

$$x = a = \tfrac{1}{2} \,, \quad y = b = \tfrac{1}{2\sqrt{3}} \,, \quad z = c = \tfrac{1}{2\sqrt{6}}$$

and $MP_1^2 = MP_5^2$, so

$$x^2 + y^2 + z^2 + w^2 = (w - d)^2 \,,$$

and, hence,

$$w = \frac{d^2 - x^2 - y^2 - z^2}{2d} = \tfrac{1}{\sqrt{10}} \,.$$

So, we have

$$P_5 = (\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}, \tfrac{1}{2\sqrt{6}}, \tfrac{\sqrt{5}}{2\sqrt{2}}) \ \text{ and } \ M = (\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}, \tfrac{1}{2\sqrt{6}}, \tfrac{1}{\sqrt{10}}) \ \text{ and } \ MP_1 = \tfrac{\sqrt{19}}{2\sqrt{10}} \,.$$

Q1529 If x, y, z are real numbers such that

$$2x - 9y + 7z = 6 \quad \text{and} \quad 7x - 6y + 2z = 9 \,,$$

then evaluate $x^2 + y^2 - z^2$.

SOLUTION We have two equations in three unknowns, and we can solve for two unknowns in terms of the third. Taking 7 times the second equation minus twice the first, and then 3 times the second minus twice the first gives

$$45x - 24y = 51 \quad \text{and} \quad 17x - 8z = 15\ ;$$

therefore,

$$y = \frac{15x - 17}{8}\ , \quad z = \frac{17x - 15}{8}$$

and we have

$$\begin{aligned} x^2 + y^2 - z^2 &= \frac{1}{8^2}\big(8^2x^2 + (15x - 17)^2 - (17x - 15)^2\big) \\ &= \frac{1}{8^2}\big((8^2 + 15^2 - 17^2)x^2 + 2(17 \times 15 - 15 \times 17)x + (17^2 - 15^2)\big) \\ &= 1\ . \end{aligned}$$

Q1530 Find all real numbers a, b, c such that $a < b < c$ and

$$a + b + c = 5\ , \quad a^2 + b^2 + c^2 = 15\ , \quad abc = 1\ .$$

SOLUTION First note that

$$(a + b + c)^2 - (a^2 + b^2 + c^2) = 2ab + 2ac + 2bc$$

and so

$$ab + ac + bc = \frac{(a + b + c)^2 - (a^2 + b^2 + c^2)}{2} = 5\ .$$

Therefore,

$$\begin{aligned} (x - a)(x - b)(x - c) &= x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc \\ &= x^3 - 5x^2 + 5x - 1 \\ &= (x - 1)(x^2 - 4x + 1) \\ &= (x - 1)(x - (2 - \sqrt{3}))(x - (2 + \sqrt{3}))\ ; \end{aligned}$$

the numbers a, b, c are the roots of this cubic: 1 and $2 - \sqrt{3}$ and $2 + \sqrt{3}$. Since we are given that $a < b < c$, we have the only solution

$$a = 2 - \sqrt{3}\ , \quad b = 1\ , \quad c = 2 + \sqrt{3}\ .$$