**Parabola Volume 53, Issue 3 (2017)**

# Constructions of a Regular Pentagon Inscribed in a Given Circle

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We sometimes encounter the problem to inscribe a regular pentagon in a given circle. Many of us have learnt only one possible construction by heart without trying to justify it. The aim of this paper is to alter this bleak reality: to present several constructions of a regular pentagon inscribed in a given circle together with proofs of their correctness.

The given circle is denoted by $k$, its center by $S$, and its radius by $r$. A regular pentagon inscribed in $k$ is called $ABCDE$. The distance between two points $X$ and $Y$ is denoted by $|XY|$.

## Finding the side length of a regular pentagon

To verify some constructions mentioned below, we need to express the side length $a_5$ of $ABCDE$ as a function of $r$.

Let $N$ be the point which is symmetric to the point $D$ with respect to $S$. Let $Q$ denote the intersection of $SN$ and $AB$ and $R$ denote the point symmetric to the point $N$ with respect to $Q$. Then the triangle $ARS$ is isosceles and $|AN| = |AR| = |SR| = a_{10}$, where $a_{10}$ is the side length of a regular decagon inscribed in $k$ (see Figure 1).

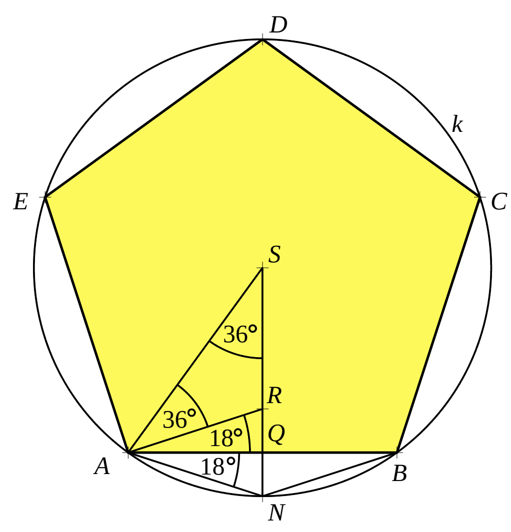


**Figure 1.** The side length of a regular pentagon

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From the similarity of triangles $SAN$ and $ANR$, we have

$$\frac{|SA|}{|AN|} = \frac{|AN|}{|NR|}\,.$$

Therefore,

$$\frac{r}{a_{10}} = \frac{a_{10}}{r - a_{10}}\,,$$

and thus

$$a_{10}^2 + ra_{10} - r^2 = 0.$$

Two roots of this quadratic equation in the one unknown $a_{10}$ are $\pm\frac{r}{2}(\sqrt{5}-1)$. Nevertheless, a side length must be a positive number, so

$$a_{10} = \frac{r}{2}(\sqrt{5}-1)\,.$$

Since $AQR$ is a right triangle, it is obvious that

$$\begin{aligned}
\frac{1}{2}a_5 = |AQ| &= \sqrt{|AR|^2 - |RQ|^2} \\
&= \sqrt{a_{10}^2 - \left(\frac{r - a_{10}}{2}\right)^2} \\
&= \frac{1}{2}\sqrt{3a_{10}^2 - r^2 + 2a_{10}r} \\
&= \frac{1}{2}\sqrt{3\left(\frac{r}{2}(\sqrt{5}-1)\right)^2 - r^2 + 2\frac{r}{2}(\sqrt{5}-1)r} \\
&= \frac{r}{4}\sqrt{3(\sqrt{5}-1)^2 - 4 + 4(\sqrt{5}-1)} \\
&= \frac{r}{4}\sqrt{10 - 2\sqrt{5}}\,.
\end{aligned}$$

Thus, the side length of a regular pentagon inscribed in a circle with radius $r$ is

$$a_5 = \frac{r}{2}\sqrt{10 - 2\sqrt{5}}\,.$$

# First construction

Now, we proceed with the above-mentioned constructions. They will be considered as finished if a line segment of length $a_5$ is drawn.

Probably, the best known construction of a regular pentagon inscribed in a given circle (see Figure 2) was given by Ptolemy (c. 100–c. 170), the mathematician, astronomer, and geographer of Greek descent, in the 2nd century in the *Almagest*, one of the most famous scientific texts of all time.[2]

[2] There exist at least two English translations of the *Almagest* which have been published. The probably more accessible one is complemented by annotations and is due to the British historian of astronomy and mathematics Gerald James Toomer; see [4].

*Let $k$ be a given circle with center $S$ and let $KL$, $MN$ be perpendicular diameters of $k$. Let $O$ be the midpoint of the line segment $KS$. If the circle with center $O$ and radius $|OM|$ intersects the line segment $KL$ at $P$, then the side length of a regular pentagon inscribed in the circle $k$ is equal to $|MP|$.*

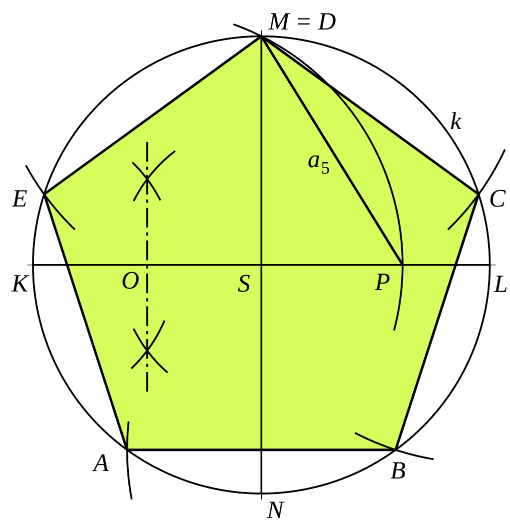


**Figure 2.** First construction

*Verification of the construction:* The length of the hypotenuse $OM$ in the right triangle $OSM$ is

$$|OM| = \sqrt{\left(\frac{r}{2}\right)^2 + r^2} = \frac{r}{2}\sqrt{5}\,.$$

Because $|OM| = |OP|$, we have[3]

$$|SP| = \frac{r}{2}\sqrt{5} - \frac{r}{2} = \frac{r}{2}(\sqrt{5} - 1)\,.$$

It follows that the length of the hypotenuse $MP$ in the right triangle $MSP$ is

$$|MP| = \sqrt{r^2 + \left(\frac{r}{2}(\sqrt{5} - 1)\right)^2} = \sqrt{r^2 + \frac{r^2}{4}(6 - 2\sqrt{5})} = \frac{r}{2}\sqrt{10 - 2\sqrt{5}}\,.$$

This result is in agreement with the side length $a_5$ of a regular pentagon that we calculated earlier. $\square$

[3]The formula implies that $|SP| = a_{10}$.

## Second construction

A perhaps faster construction of a regular pentagon inscribed in a circle is the following one (see Figure 3):

*Let $k$ be a given circle with center $S$ and let $KL$, $MN$ be perpendicular diameters of $k$. Let $O'$ be a intersection of $k$ and the circle with center $K$ and radius $|KS|$. If the circle with center $O'$ and radius $|KN|$ intersects the diameter $KL$ at $P'$, then the side length of a regular pentagon inscribed in $k$ is equal to $|MP'|$.*

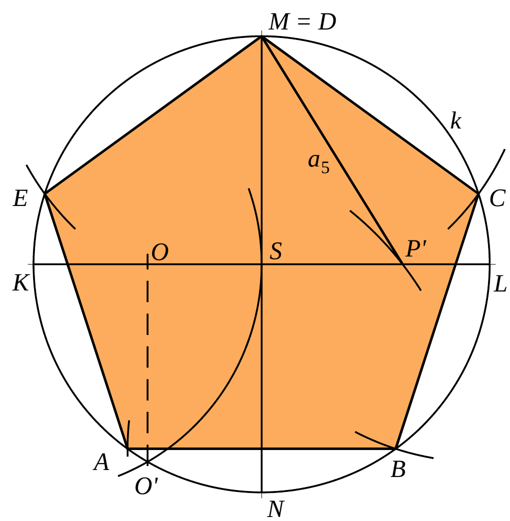


**Figure 3.** Second construction

*Verification of the construction:* Let $O$ denote the foot of the perpendicular from $O'$ to $KL$. The line segment $OO'$ is the altitude of an equilateral triangle with the side length $r$; therefore, $|OO'| = \frac{r}{2}\sqrt{3}$. Since the hypotenuse in an isosceles right triangle with leg lengths $r$ has length $r\sqrt{2}$, we obtain $|O'P'| = |KN| = r\sqrt{2}$. According to the Pythagorean Theorem, $|OP'| = \sqrt{2r^2 - \frac{3}{4}r^2} = \frac{r}{2}\sqrt{5}$. Thus, the point $P'$ is the same as the point $P$ from the first construction. This finishes the proof. □

## Third construction

The following construction (see Figure 4) was discovered by the amateur mathematician Yosifusa Hirano. It was presented in the manuscript *Sanpo Jyojutu Kaigi* [*Solutions to Sanpo Jyojutu*] which was written by Hirano's friend Chorin Kawakita (1840–1919). It is contained in the book *Japanese Temple Geometry Problems* [1] which was published by Hidetoshi Fukagawa and Daniel Pedoe in 1989.

This construction also starts with drawing perpendicular diameters $KL$, $MN$ of $k$ and also the midpoint $O$ of the line segment $KS$. Then a regular pentagon may be constructed only by compass.

*Let $k$ be a given circle with center $S$ and let $KL$, $MN$ be two perpendicular diameters of $k$. Draw the midpoint $O$ of the line segment $KS$ and then the line segment $ON$. Construct the circle with center $O$ and radius $|OS|$ and denote the point where this circle crosses the line segment $ON$ by $T$. Draw the circle with center $N$ and radius $|TN|$. Intersections $A$, $B$ of the latter circle and the circle $k$ are the endpoints of the side of a regular pentagon inscribed in $k$.*

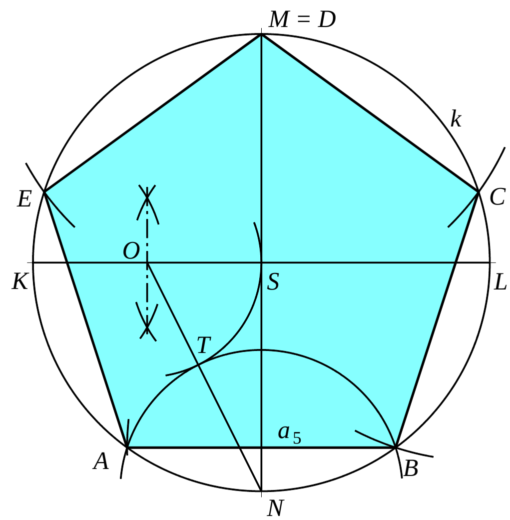


**Figure 4.** Third construction

*Verification of the construction:* It is obvious that the length of the hypotenuse $ON$ in the right triangle $ONS$ is

$$|ON| = \sqrt{r^2 + \left(\frac{r}{2}\right)^2} = \frac{r}{2}\sqrt{5}\,.$$

Clearly, $|NT| = \frac{r}{2}\sqrt{5} - \frac{r}{2} = \frac{r}{2}(\sqrt{5} - 1)$, and hence (see above) $|NT| = |NA| = |NB|$ is the side length $a_{10}$ of a regular decagon inscribed in $k$ with radius $r$. It follows that the line segment $AB$ is a side of a regular pentagon inscribed in $k$. □

# Fourth construction

The fourth construction, given by the English mathematician Herbert William Richmond (1863–1948) in the 1893 paper *A construction for a regular polygon of seventeen sides* [3], also begins with drawing perpendicular diameters $KL$, $MN$ of $k$ and finding the midpoint $O$ of the line segment $KS$ (see Figure 5).

*Let $k$ be a given circle with center $S$ and let $KL$, $MN$ be two perpendicular diameters of $k$. Find the midpoint $O$ of the line segment $KS$ and draw the line segment $OM$. Next, draw a bisector $o$ of the acute angle $SOM$ and denote its intersection with $MN$ by $U$. Through this point $U$ draw a line $p$ parallel to $KL$ and denote one of its intersections with $k$ by $V$. Then the side length of a regular pentagon inscribed in $k$ is equal to $|VM|$.*

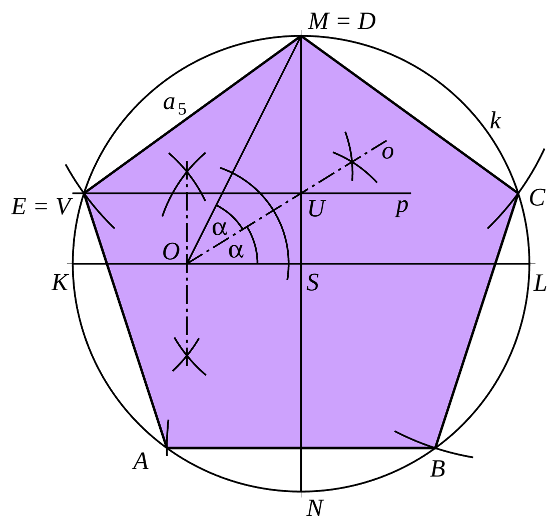


**Figure 5.** Fourth construction

*Verification of the construction:* If $\alpha$ is the size of the acute angle $SOU$, then the size of the acute angle $SOM$ is equal to $2\alpha$. Clearly, $\tan 2\alpha = 2$. The well-known trigonometric identity

$$\tan 2x = \frac{2\,\tan x}{1 - \tan^2 x}$$

implies that

$$2 = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

and, consequently,

$$2\,\tan^2 \alpha + 2\,\tan \alpha - 2 = 0\,.$$

Substituting $a = \tan \alpha$ yields

$$a^2 + a - 1 = 0\,.$$

Although this quadratic equation in the one unknown $a$ has two real roots $\frac{1}{2}(-1 \pm \sqrt{5})$, only the positive number $\frac{1}{2}(-1 + \sqrt{5})$ can be equal to $\tan \alpha$, $\alpha \in (0;\, 45°)$. Thus,

$$a = \frac{-1 + \sqrt{5}}{2} = \tan \alpha = \frac{|US|}{\frac{r}{2}}$$

and, after rearranging, the following length is obtained:

$$|US| = \frac{r}{4}(\sqrt{5} - 1)\,.$$

The length $|VM|$ can be again calculated using the Pythagorean Theorem: first, we can calculate the length $|UV|$ of the leg in the right triangle $VSU$ and then compute the length $|VM|$ of the hypotenuse in the right triangle $MVU$. However, let us use analytic geometry for a change.

If we set up a Cartesian coordinate system with $S$ at the origin and the diameter $KL$ on the $x$-axis (see Figure 6), then $V$ has coordinates $\left[x;\ \frac{r}{4}(\sqrt{5}-1)\right]$ and lies on the circle whose equation is $x^2+y^2=r^2$. Hence,

$$x^2+\frac{r^2}{16}(6-2\sqrt{5})=r^2$$

and, consequently,

$$x^2=\frac{r^2}{16}(10+2\sqrt{5})\,.$$

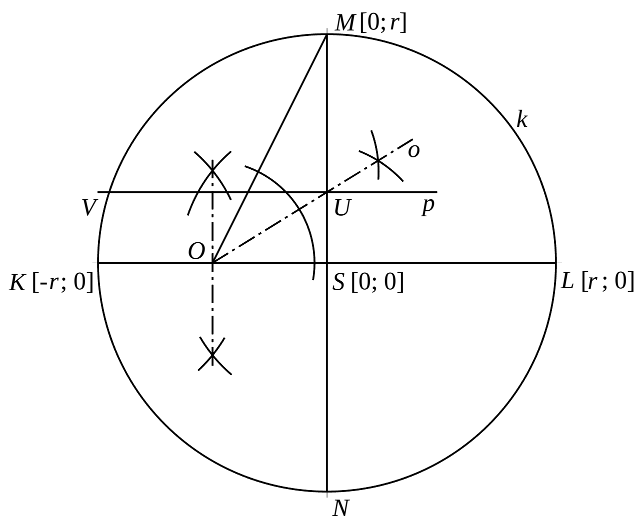


**Figure 6.** Verification of the construction

If we adhere to Figures 5 and 6, respectively, then we see that $x=-\frac{r}{4}\sqrt{10+2\sqrt{5}}$. Finally, we calculate the distance between the points $V\left[-\frac{r}{4}\sqrt{10+2\sqrt{5}};\ \frac{r}{4}(\sqrt{5}-1)\right]$ and $M\,[0;\ r]$:

$$\begin{aligned}|VM|&=\sqrt{\left(\frac{r}{4}\sqrt{10+2\sqrt{5}}\right)^2+\left(r-\frac{r}{4}(\sqrt{5}-1)\right)^2}\\&=\sqrt{\frac{r^2\left(10+2\sqrt{5}\right)}{16}+\frac{r^2(30-10\sqrt{5})}{16}}\\&=\frac{r}{4}\sqrt{40-8\sqrt{5}}\\&=\frac{r}{2}\sqrt{10-2\sqrt{5}}\,.\end{aligned}$$

Therefore, the fourth construction is also correct. □

## Fifth construction

The last construction (see Figure 7) was published by David Nelson without proof in the 1977 paper *A regular pentagon construction* [2]. Our proof is given below.

*Let $k$ be a given circle with center $S$ and radius $r$ and let $KLM$ be an isosceles triangle inscribed in $k$ with the base line $LM$ and with the altitude from the point $K$ of the length $h = \frac{5}{4}r$. Find the points $F$ and $G$ on $k$ such that $|\angle KSF| = |\angle KSG| = 60°$. Denote the intersection of the line segments $SF$ and $KL$ by $H$, and of the line segments $SG$ and $KM$ by $I$. Draw the line $HI$ and denote one of its intersections with $k$ by $V'$. Then the side length of a regular pentagon inscribed in $k$ is equal to $|KV'|$.*

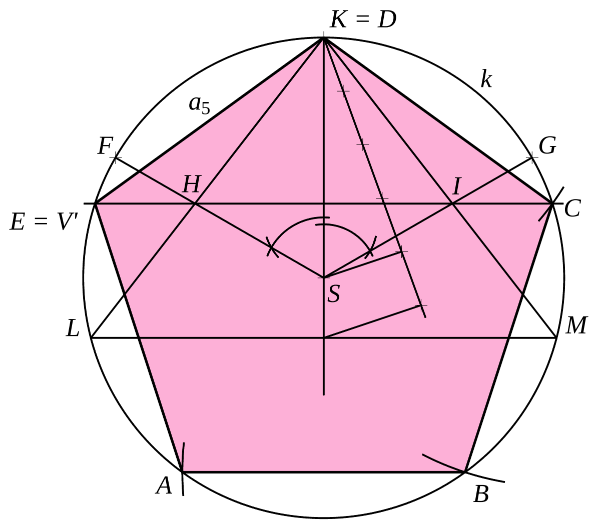


**Figure 7.** Fifth construction

*Verification of the construction:* Let $J$ and $U'$ be the intersections of the line $KS$ with the lines $LM$ and $HI$, respectively (see Figure 8). Through the point $S$ construct a line parallel to the line $HI$ and denote its intersection with the line segment $KL$ by $T$. Next, denote the foot of the perpendicular dropping from the point $H$ to the line $TS$ by $W$.

Since $|SL| = r$ and $|SJ| = \frac{r}{4}$, we have $|LJ| = \sqrt{r^2 - \left(\frac{r}{4}\right)^2} = \frac{r}{4}\sqrt{15}$. From the similarity of triangles $TSK$ and $LJK$, we obtain

$$\frac{|TS|}{|SK|} = \frac{|LJ|}{|JK|}$$

and, consequently,

$$|TS| = \frac{|LJ| \cdot |SK|}{|JK|} = \frac{\frac{r}{4}\sqrt{15} \cdot r}{\frac{5}{4}r} = \frac{r}{5}\sqrt{15}\,.$$

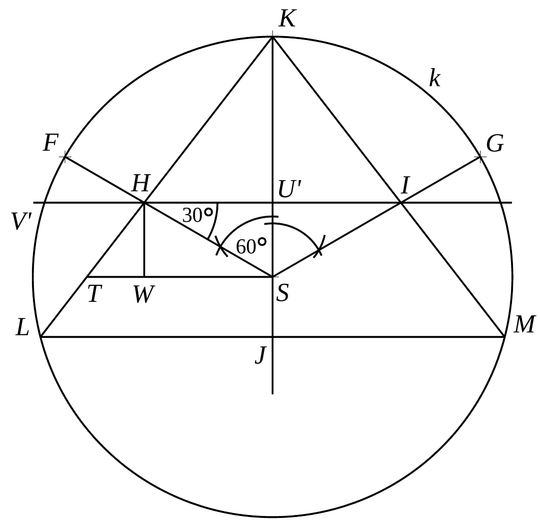


**Figure 8.** Verification of the construction

Furthermore, we have

$$\tan 60^\circ = \sqrt{3} = \frac{|HU'|}{|U'S|}\,,$$

which gives us $|HU'| = \sqrt{3}\,|U'S|$. Triangles $TWH$ and $TSK$ are similar. It follows that

$$\frac{|HW|}{|WT|} = \frac{|KS|}{|ST|}\,,$$

and, hence,

$$\begin{aligned} |U'S| = |HW| &= \frac{|KS|}{|TS|}\,(|TS| - |HU'|) \\ &= \frac{r}{\frac{r}{5}\sqrt{15}}\left(\frac{r}{5}\sqrt{15} - \sqrt{3}\,|U'S|\right) = r - \sqrt{5}\,|U'S|\;. \end{aligned}$$

Thus,

$$|U'S| = \frac{r}{1+\sqrt{5}} = \frac{r}{4}(\sqrt{5}-1)$$

and the line $HI$ plays the role of the line $p$ from the fourth construction, which completes the last proof. □

# Conclusion

Various constructions of a regular pentagon can be found on several websites; see for example the well-known encyclopedia *Cut The Knot* [5]. We wish the readers good luck in discovering other possible constructions.

## References


[1] H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, Charles Babbage Research Centre, Winnipeg, 1989.

[2] D. Nelson, A regular pentagon construction, *Math. Gaz.* **61** (1977), 215–216.

[3] H. W. Richmond, A construction for a regular polygon of seventeen sides, *Quart. J. Pure Appl. Math.* **26** (1893), 206–207.

[4] G. J. Toomer, *Ptolemy's Almagest*, second edition, Princeton University Press, Princeton, New Jersey, 1998.

[5] A. Bogomolny, *Interactive Mathematics Miscellany and Puzzles*, 1997–2017, http://cut-the-knot.org/.