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Making Collatz Cry:
An Examination of Necessarily-Failing Collatz-Like Conjectures

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1 Introduction

The traditional Collatz Conjecture states that, for any number, if you divide by 2 if the number is even and, if odd, then multiply by 3 and add 1, and repeat, you will eventually reach 1. For example, starting with 5, we have $3 \times 5 + 1 = 16$, $16 \div 2 = 8$, $8 \div 2 = 4$, $4 \div 2 = 2$, and finally $2 \div 2 = 1$. This conjecture has proven un-provable to mathematicians for 80 years now, with many believing that new areas of mathematics must first be invented to solve the problem. In this paper, we explore Collatz-like rulesets and we examine which must necessarily succeed and which must necessarily fail. We first define some important words:

Ruleset A set of mathematical operations that is applied iteratively to successive integers based on their remainder modulo a certain constant. The traditional Collatz ruleset states that any even integers should be divided by two and odd integers should be multiplied by three and one added.

Succeed A ruleset is said to *succeed* if all integers, when inputted into the ruleset, must reduce to 1. We will not explore what happens after the number 1 is achieved (in the traditional Collatz Conjecture it cycles 1, 4, 2, 1 infinitely).

Cases The "cases" of a ruleset signifies the number of rules that the ruleset contains. The traditional Collatz Conjecture ruleset can be said to be in 2 cases.

For the purposes of this paper, we will explore what happens when a Collatz-like ruleset is created in a cases, where a is an integer. Motivated by a desire to ensure that our rulesets are as close as possible to the classical Collatz Conjecture, we require them to have two features. All rulesets must have a "primary rule": all integers expressible as an are divided by a. As an example, for a Collatz-like conjecture in 5 cases, all numbers expressible as $5n$ become n. Also, all other rules must be of the form "multiply by x and add y", where x and y are non-zero integers. There are always a rules for a ruleset in a cases. In particular, $a - 1$ of these rules result in numbers increasing and 1 of these (the primary rule) results in a decrease.

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An example of a Collatz-like ruleset in 3 cases that must necessarily succeed is

$3n$	is divided by 3;
$3n+1$	is multiplied by 2 and 1 is added; and
$3n+2$	is multiplied by 2 and 2 is added.

This ruleset can easily be shown to be strictly decreasing for all n and therefore it must succeed. We write this in shortened notation as

$$\begin{array}{rcl} 3n & \to & n \\ 3n+1 & \to & 2(3n+1)+1 \\ 3n+2 & \to & 2(3n+2)+2 \end{array}$$

or in even shorter form as

$$3n \to n; \quad 3n+1 \to 6n+3; \quad 3n+2 \to 6n+6\,.$$

2 Collatz cycles – Collatz-like conjectures that must fail

The first thing to note is that if the ruleset is such that an integer of any form is run through a rule that causes it to return to the same form, then the ruleset must fail. For example, for a ruleset in 6 cases, numbers expressible as $6n+1$ may be multiplied by 3, and 4 is added. This is the rule $6n+1 \to 18n+7$ so this rule is applied repeatedly and numbers expressible as $6n+1$ must tend to infinity. This is an example of the most basic form of Collatz cycle, or a set of rules which continually feed into each other in such a way that any number run through them will never reach the primary rule. More complex cycles can be set up, where numbers are of the form $an+b \to an+d \to an+b$ and so numbers of both forms $an+b$ and $an+d$ must tend to infinity. For a ruleset in a cases, the largest cycle that can be made is of size $a-1$. If the primary rule is incorporated into the cycle at any stage, then the cycle does not necessarily cause a strict increase in integers in that cycle, and the primary rule may even cause the ruleset to succeed. Whether the decreasing effect of the primary rule outweighs the increasing effect of the non-primary rules is something to be shown on a case-by-case basis. Being able to show this generally would result in a proof of the Collatz Conjecture.

Theorem 1. *There are an infinite number of Collatz-like rulesets in a cases that fail for all integers other than powers of a.*

Proof. Because of the nature of the primary rule, any number that is a power of n must decrease to 1 by repeatedly applying the primary rule. However, an infinite number of Collatz cycles can be constructed such that all other numbers must tend to infinity. Let us set up a ruleset such that

$$\begin{array}{rcl} an & \to & n \\ an+1 & \to & c_1(an+1)+d_1 \\ an+2 & \to & c_2(an+2)+d_2 \\ an+3 & \to & c_3(an+3)+d_3 \end{array}$$

and so on. We wish to build a cycle such that numbers expressible as

$$an+1 \to an+2; \quad an+2 \to an+3; \quad an+3 \to an+4; \quad \cdots \quad ; an+(a-1) \to an+1$$

in order to omit the primary rule. In general,

$$an+m \to an+(m+1) \qquad \text{or} \qquad an+(a-1) \to an+1\,.$$

Therefore, we require $c_m(an+m)+d_m = c_m(an)+(m+1)$. This is possible if $d_m = m+1-c_m(m)$ for $1 < m < a-1$. For $m = a-1$, the identity $d_m = m+2-c_m(m)$ must hold to ensure that a cycle is created, omitting the primary rule and ensuring that numbers of every form are included. There are therefore an infinite number of possible rulesets that result in cycles which can be created by varying c_m and calculating the relevant d_m for each m. An example of such a cycle is

$$\begin{aligned} 4n &\to n \\ 4n+1 &\to 5(4n+1)+5 \\ 4n+2 &\to 6(4n+2)+3 \\ 4n+3 &\to 17(4n+3)+2\,. \end{aligned}$$

□

3 Random Collatz-like conjectures

If we consider Collatz-like conjectures as a series of rules that link to each other, then we can derive a very interesting result from the idea of randomly generating these conjectures. Let us introduce the notion of a "random Collatz-like ruleset": a ruleset in which for every rule of the form $an+b \to cn+d$, c and d have been randomly chosen. We can now prove the following theorem.

Theorem 2. *The fraction of random Collatz-like rulesets which succeed is at most $\frac{x!}{x^x}$, where x is the number of distinct rules in the ruleset.*

Proof. For a Collatz-like ruleset to succeed, it must not contain cycles. We must thus find the probability that a random Collatz-like ruleset with x rules will contain no cycles. To do this, we consider each ruleset as a set of interlinked nodes. Each rule links to exactly one other, and in a random Collatz-like ruleset this is determined at random. Now, for there to be no cycles, every rule must link, eventually, to the primary rule in a tree formation. We will start with an unlinked set of nodes and attempt to add links one after the other in a random way until this condition is fulfilled. The first link we add must go to the primary rule, which it will with probability $\frac{1}{x}$. (There are x rules to link to, of which only 1 will work.) The second link may go to either the primary rule, or the rule which we just linked to the primary rule. The probability that it will go to one of these is $\frac{2}{x}$. The third link may go to either of the three newly linked rules, which it will do with probability $\frac{3}{x}$, and so on. One possible hiccough will be if a rule links to an unlinked rule which then is later linked to the primary rule, but since the order in which the links are added does not affect the final outcome, we may resolve this by simply adding the same links in a different order. Now, to find the probability $P(x)$ of all links being part of this tree formation with no cycles, we multiply the probabilities for individual links to get

$$P(x) = \frac{1}{x} \times \frac{2}{x} \times \frac{3}{x} \times \cdots \times \frac{x-2}{x} \times \frac{x-1}{x} = \frac{(x-1)!}{x^{x-1}}.$$

Multiply above and below by x to get the more elegant $P(x) = \frac{x!}{x^x}$. □

The $P(x)$ value falls to zero extremely quickly as x becomes larger:

$$P(2) = \frac{1}{2}$$
$$P(3) = \frac{2}{9}$$
$$P(4) = \frac{3}{32}$$
. . . and so on.

Should the classical Collatz Conjecture be true, then it will be part of a very rare species.