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Problems 1541–1550

Q1541 Consider the equation

$$29x + 30y + 31z = 366\ , \qquad (*)$$

where x, y, z are positive integers with $x < y < z$.

(a) Without any writing or computer assistance, find x, y, z which satisfy these conditions.

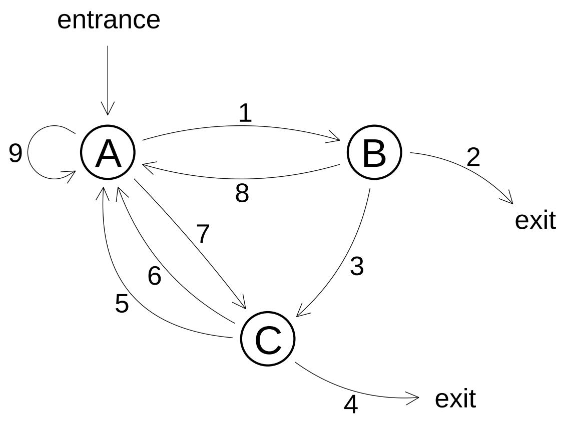
(b) Prove that your solution from (a) is the only possibility.

Q1542 Solve the equation

$$x = \sqrt{2017 + \sqrt{2017 + x}}\ .$$

Q1543 Let ABC be a triangle with $\angle ABC = 120°$, and let P be the circumcentre of the triangle, that is, the point for which the lengths AP, BP and CP are all equal. Prove that the ratio of lengths AC/AP is equal to $\sqrt{3}$.

Q1544 To reach an exit of the MessConnex tollway, drivers have to get through a system of three roundabouts, shown as A, B and C in the diagram.



Each roundabout has three outgoing roads; two of the nine roads lead out of the MessConnex; the others stay within the system, one of them even returning to the very same roundabout. There are no signs to indicate the correct exit, so the drivers just have to guess; and all the exits look identical, so if drivers return to the same roundabout, they just have to guess again.

The numbers shown on the outgoing roads are the toll (in dollars) charged for using each road. Clearly, a lucky driver could get out of the system for \$3; but most drivers would have to spend much more. How much would it cost the average driver?

Q1545 The numbers $1, 2, 3, \ldots, mn$ are arranged in an array of m rows and n columns in such a way that each of the m rows has the same sum, and each of the n columns has the same sum. Prove that m and m are either both even, or both odd.

Q1546 (a) Let n, a, b be positive integers such that

$$n^2 < a < b < (n+1)^2 \ .$$

Prove that ab cannot be a perfect square.

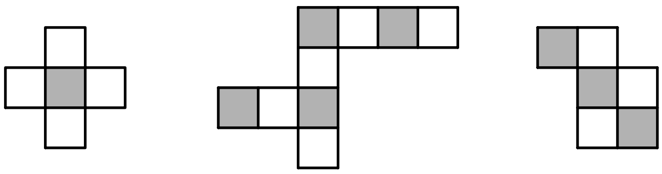
(b) Find infinitely many examples of positive integers n, a, b, c such that

$$n^3 < a < b < c < (n+1)^3$$

and abc is a perfect cube.

Q1547 If the points $A, B, C, D, E, F, G, H, I$ all lie in a plane, and ABC is an equilateral triangle with side length 1, and $BCDE$ is a square, and $BEFGHI$ is a regular hexagon, find the distance AG.

Q1548 A *polyomino* is a figure consisting of unit squares joined along their edges. Every join must involve the full edge of both squares. We can give a polyomino a "chessboard" colouring (alternately light and dark) and calculate the ratio of light to dark squares. The diagram shows some possibilities with ratios $\frac{4}{1} = 4$, $\frac{5}{4}$ and $\frac{3}{3} = 1$.



Prove that if $\frac{m}{n}$ is a rational number between $\frac{1}{3}$ and 3, then there is a chessboard–coloured polyomino such that the ratio of light to dark squares equals $\frac{m}{n}$.

Q1549 Let a, b be positive integers with no common factor, and let n be an integer, $n \geq 2$. Prove that $a^{n-1} + b^{n-1}$ is not a factor of $a^n + b^n$.

Q1550 (a) Let $f(x)$ be a polynomial with integral coefficients which has four different integer roots. Prove that there is no integer a such that $f(a)$ is prime.

(b) Find an example of a polynomial $f(x)$ with integral coefficients and three different integer roots, and a value of a such that $f(a)$ is prime.