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Solutions 1531–1540

Q1531 Take any four consecutive whole numbers, multiply them together and add 1. Make a conjecture and prove it!

SOLUTION The resulting number can, for instance, be expressed as a square:

$$\begin{aligned}(n-1)n(n+1)(n+2)+1 &= (n-1)(n+2)n(n+1)+1\\ &= \big((n^2+n-1)-1\big)\big((n^2+n-1)+1\big)+1\\ &= (n^2+n-1)^2\,.\end{aligned}$$

Q1532 Let ABC be a triangle with longest side BC and let P be a point in the interior of the triangle. Show that $AP < BP + PC$.

SOLUTION Note that $BP + PC > BC > \max\{AC, AB\} > AP$.

Q1533

(a) Show that if p is odd, then $x^p - 1 = (x-1)(x^{p-1} + x^{p-2} + \cdots + x + 1)$.

(b) Hence, show that if x is odd, then the highest power of 2 which divides $x^p - 1$ also divides $x - 1$.

(c) Find the highest power of 2 which divides $1999^{2000} - 1$.

SOLUTION

(a) Expand the right-hand side of the equation.

(b) If x is odd, then $(x^{p-1} + x^{p-2} + \cdots + x + 1)$ is odd since it is the sum of an odd number of odd numbers.

(c) $1999^{2000} - 1 = ((1999^{16})^{125} - 1)$ so, by (b), the highest power of 2 that we seek is the highest power of 2 dividing $1999^{16} - 1$. This number is

$$(1999^8+1)(1999^4+1)(1999^2+1)(1999+1)(1999-1)\,.$$

The first term is divisible by 2 but not 4; this is easy to see if calculating modulo 8. This is also true for each of the other terms. So the power of 2 is 2^8.

Q1534 Let $ABCD$ be a convex quadrilateral and let P, Q, R, S be points on AB, BC, CD, DA respectively, such that $AP = \frac{1}{4}AB$, $QC = \frac{1}{4}BC$, $CR = \frac{1}{4}CD$ and $SA = \frac{1}{4}DA$.

(a) Show that $PQRS$ is a parallelogram.

(b) Find the ratio of the area of $PQRS$ to that of $ABCD$.

SOLUTION

(a) This is easily shown by similarity.

(b) $PQRS = \frac{3}{8} ABCD$.

Q1535 Find all positive integer solutions to $2x^2 - 2xy + y^2 = 65$.

SOLUTION We can rewrite the equation as $(x - y)^2 + x^2 = 65$. The only two square integer pairs that add up to 65 are $1 = 1^2$ and $64 = 8^2$, and $16 = 4^2$ and $49 = 7^2$, and these pairs in reverse order. The solutions (x, y) are thus

$$(8, 9)\,,\ (1, 9)\,,\ (7, 3)\,,\ (4, 7)\,.$$

Q1536 The base notation a_b appearing in this problem is mostly recently explained in the *Parabola* article here.

(a) Show that whatever base b is used, the number $(21)_b$ is never equal to twice $(12)_b$.

(b) Find all the numbers and all bases $b \leq 12$ for which there exists a two digit number $(ac)_b$ which is twice the number obtained by reversing its digits.

(c) Find all bases b and all numbers $n = (ac)_b$ such that $n = 2 \times (ac)_b$.

SOLUTION

(a) Now, $(21)_b = 2b + 1$ and $(12)_b = b + 2$; however, $2(b + 2)$ is even and cannot equal $2b + 1$.

(b) By trial and error, we find $b = 5$, $(ac)_b = (31)_b$, $b = 8$, $(ac)_b = (52)_b$, and $b = 11$, $(ac)_b = (73)_b$.

(c) The condition is $ab + c = 2(cb + a)$, or $a(b - 2) = c(2b - 1)$. Now, if $d|(b - 2)$ and $d|(2b - 1)$, then we have $d|(b - 2) - 2(2b - 1)$ and so $d|3$; hence, $d = 1$ or $d = 3$. If $d = 1$, then $a = 2b - 1$ which is greater than b. If $d = 3$, then we can write $b - 2 = 3k$ which implies that $ka = (2k + 1)c$ and, since $\gcd(k, 2k + 1) = 1$, the only solution with $a, c < b$ is $c = k$ and $a = 2k + 1$. Hence, the required conditions are $b = 3k + 2, c = k, a = 2k + 1$.

Q1537 Denote the top of a cube by $ABCD$ and the bottom by A_1, B_1, C_1, D_1, so that A is directly above A_1 and so on. Take midpoints of the six edges AB, BB_1, B_1C_1, C_1D_1, D_1D and DA. Show that a plane containing any three of these points contains them all and deduce that these points form the vertices of a regular hexagon.

SOLUTION Let P be the midpoint of AA_1, S the midpoint of CC_1, U the midpoint of AB and T the midpoint of BC. Extend B_1B to X such that $BX = AP$. The rest of the solution is straight-forward.

Q1538

(a) Simplify $(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$.

(b) Show that $(a^2+b^2+c^2-ab-ac-bc) \geq 0$.

(c) Prove that if x, y, z are positive real numbers, then $\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$.

SOLUTION

(a) $a^3+b^3+c^3-3abc$

(b) $$\begin{aligned} a^2+b^2+c^2-ab-ac-bc &= \frac{1}{2}\left((a^2+b^2-2ab)+(a^2+c^2-2ac)+(b^2+c^2-2bc)\right) \\ &= \frac{1}{2}\left((a-b)^2+(a-c)^2+(b-c)^2\right) \\ &\geq 0\,. \end{aligned}$$

(c) By parts (a) and (b), $x^3+y^3+z^3-3xyz \geq 0$. Since the function $f(u)=u^3$ is convex for positive u, we have that

$$\frac{x+y+z}{3} = \sqrt[3]{\left(\frac{x+y+z}{3}\right)^3} \geq \sqrt[3]{\frac{x^3+y^3+z^3}{3}} \geq \sqrt[3]{xyz}\,.$$

Q1539 If we expand $(2+x)^{18}$ as a polynomial, then we obtain

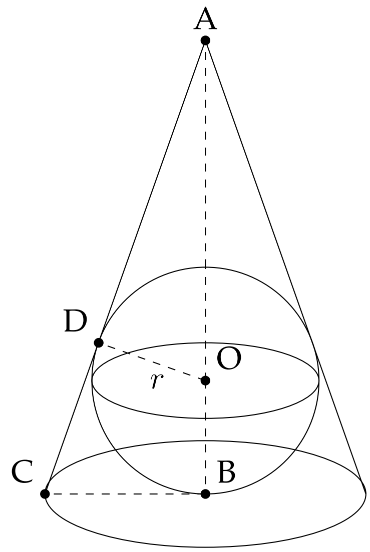
$$(2+x)^{18} = a_0 + a_1 x + a_2 x^2 + \cdots + a_{18} x^{18}\,,$$

where $a_0, a_1, \ldots, a_{18}$ are integers.
Without using the Binomial Theorem, find a_0, a_1, a_{18} and $a_0 + a_1 + \cdots + a_{18}$.

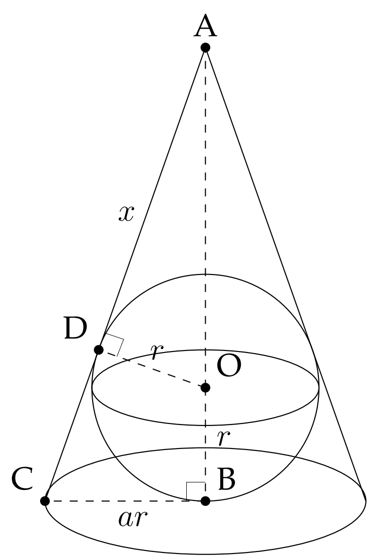
SOLUTION Write out the product as $\underbrace{(2+x)(2+x)\cdots(2+x)}_{18 \text{ times}}$.

Then clearly, $a_0 = 2^{18}, a_{18} = 1$ and the term with one x can be obtained from 18 terms each with a coefficient of 2^{17}, so $a_1 = 18 \times 2^{17}$. To get the sum of the coefficients, set $x=1$; this gives 3^{18}.

Q1540 Circumscribe a right circular cone about a sphere of radius r such that the cone has minimum volume. Prove that the cone has exactly double the volume of the sphere.



SOLUTION Since the radius of the cone must be greater than the radius of the sphere, let the cone's radius be ar, such that $a > 1$ and ar is the volume-minimizing radius of the cone. In addition, let $\overline{AD} = x$.



Using the fact that $\triangle ADO \sim \triangle ABC$, we can write the equation

$$\frac{x}{r} = \frac{\sqrt{x^2 + r^2} + r}{ar}.$$

We solve for x in terms of a and r, first getting $(ax-r)^2 = x^2 + r^2$ and thus $x = \dfrac{2ar}{a^2-1}$. The height of the cone is

$$\sqrt{\left(\frac{2ar}{a^2-1}\right)^2 + r^2} + r = \sqrt{\frac{4a^2r^2 + r^2(a^2-1)^2}{(a^2-1)^2}} + r = \frac{r(a^2+1)}{a^2-1} + r\,, \tag{1}$$

so the volume of the cone, expressed as a function $f(a)$ in a, is

$$f(a) = \frac{1}{3}\pi(ar)^2\left(\frac{r(a^2+1)}{a^2-1} + r\right).$$

Because the cone must have minimum volume, we must find the value of a that minimizes our function, so we calculate the derivative of our volume function:

$$f'(a) = \frac{\pi r^3}{3}\frac{4a^3(a^2-2)}{(a^2-1)^2}\,.$$

Setting this derivative equal to 0 and solving for a, we find that $a = 0, \pm\sqrt{2}$. Since $a > 1$, the correct solution is $a = \sqrt{2}$. Thus, the radius of the cone is $r\sqrt{2}$ and, by (1), the height of the cone is $4r$. Therefore, the volume of the cone is

$$\frac{1}{3}\pi(r\sqrt{2})^2 4r = \frac{8}{3}\pi r^3\,.$$

Since the volume of the sphere is $\frac{4}{3}\pi r^3$, the volume of the cone is exactly double the volume of the sphere, as claimed.