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Terminating sum of digits of a positive integer

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Dedicated to Eleanor for restoring my hope in humanity.

The terminating sum $T(n)$ of a positive integer n is obtained by repeatedly adding the digits of n until a single digit number is obtained. For instance, $n = 2018^{2018}$ is a number with 6670 digits, whose sum is 30001. Since $4 = 3 + 1$, $T(n) = T(2018^{2018}) = 4$. The web references [1] and [2] list $T(n^m)$ for large positive integers n and m, obtained by using MATLAB, C++ and Python. By extending the divisibility rule of 9, this note provides the following closed form expression for $T(n)$.

Theorem. *Let n be a positive integer. Then*

$$T(n) = \begin{cases} 9 & \text{if } n \equiv 0 \pmod 9 \\ n \mod 9 & \text{if } n \not\equiv 0 \pmod 9 \end{cases}$$

By using this theorem, we only need to add the digits of n once in order to calculate $T(n)$.

Example. *The number $n = 99$ has digit sum 18 which in turn has digit sum 9, so $T(99) = 9$. This is also directly given by the theorem since $n = 99 \equiv 0 \pmod 9$.*

In contrast, $n = 98 \equiv 8 \pmod 9$, so $T(98) = 8$ by the theorem.

Proof. Let

$$n = \sum_{k=0}^{m} d_k 10^k = \sum_{k=0}^{m} d_k + \sum_{k=0}^{m-1} d_k(10^k - 1)$$

where $d_0, d_1, \ldots, d_m$ are the digits of n. Let $S(n)$ be the sum of the digits of n. That is,

$$S(n) = \sum_{k=0}^{m} d_k \,.$$

Since $10^k - 1$ is an integer consisting of digits of 9 only, $d_k(10^k - 1)$ is divisible by 9, so we conclude that

$$n \equiv S(n) \pmod 9 \,.$$

By repeatedly applying the above procedure, we obtain the required result. □

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The theorem can be extended to n^m for positive integers n and m in the table below.

$n \pmod 9$	k	$m \pmod k / T(n^m)$
0	10	$0/1$, $m \geq 1/9$
1	10	All$m/1$
2	6	$0/1$, $1/2$, $2/4$, $3/8$, $4/7$, $5/5$
3	10	$0/1$, $1/3$, $m \geq 2/9$
4	3	$0/1$, $1/4$, $2/7$
5	6	$0/1$, $1/5$, $2/7$, $3/8$, $4/4$, $5/2$
6	10	$0/1$, $1/6$, $m \geq 2/9$
7	3	$0/1$, $1/7$, $2/4$
8	10	$0/1$, $1/6$, $m \geq 2/9$

Closing remarks

There could be those who might question the value of these results such as these. They may be less important than an accounting ledger. To a mathematics devotee, their mind is a solitaire game in search for the first step to climb the rainbow.

References

[1] www.quora.com, How do I find sum of digits of a large no raised to a large power, last retrieved 27 April 2018.

[2] www.quora.com, How do I find the sum of the digits of really large numbers with MATLAB or another language. Example: 3^600 has 287 digits. How do I add them?, last retrieved 27 April 2018.