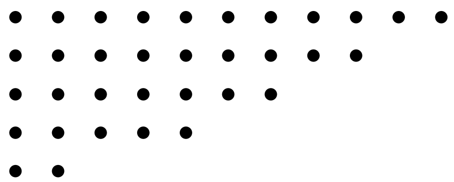
Parabola Volume 54, Issue 1 (2018)

Problems 1551–1560

Q1551 We have a pattern of 34 dots arranged as shown.



It is permitted to remove any three dots, provided that one of them is exactly midway between the other two (the three dots may form a line in any direction – horizontal, vertical, diagonal or oblique); then to remove another three dots under the same condition; and so on. If we remove 33 dots, which are the possibilities for the remaining dot?

Q1552 Pat and Sandy are sharing a project. Pat works for 12 days and completes more than half the job; Sandy then takes over and finishes the job, taking another 12 days. If, instead, Pat had done exactly half the job before Sandy took over for the rest, and if they both worked at the same rate as in the previous scenario, then the whole project would have taken 25 days. How many days would Pat have worked for?

Q1553 For every positive integer n we define a positive integer $f(n)$ such that the following property is true:

$$\text{if } m, n \text{ are any positive integers, then } m^2 + f(n) \mid mf(m) + n,$$

where the notation $a \mid b$ denotes that a is a factor of b. It is easy to see that $f(n) = n$ for all n is one possibility; prove that there is no other possibility.

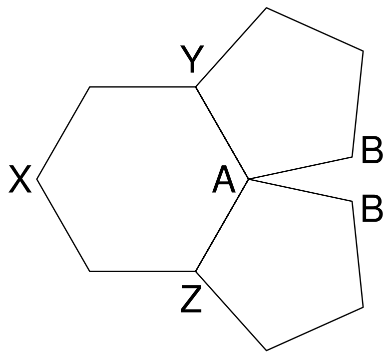
Q1554 Suppose that it is possible to arrange the numbers $1, 2, 3, \ldots, mn$ in an array of m rows and n columns in such a way that each of the rows has the same sum, and each of the columns has the same sum. (Compare with Problem 1545, the solution of which appears in the previous issue). Show that it is possible to arrange the numbers $1, 2, 3, \ldots, m^2n$ in an array of m rows and mn columns with the same property.

Q1555 Show that if the quadratic equation

$$x^2 - px + p = 0$$

has two (real) solutions x_1 and x_2, then $x_1^2 + x_2^2 > 2(x_1 + x_2)$.

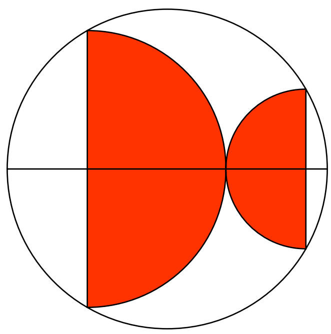
Q1556 A shape consisting of a regular hexagon and two regular pentagons is cut out of cardboard; the pentagons are bent upwards along the lines AY and AZ until the two points marked B meet. What is then the angle $\angle XAB$?



Q1557 Eric is playing a game in which he rolls a (normal, six–sided) die three times, and wins if his three rolls are all different and in increasing order. For example, $1, 4, 5$ wins, but $4, 1, 5$ loses, and so does $4, 5, 5$. In the middle of the game Eric calls you on the phone and tells you that his second roll was bigger than his first. If the game continues, what is Eric's chance of winning?

Q1558 Let n and k be positive integers with $n \geq 2k - 1$. In how many ways can k numbers be selected from $\{ 1, 2, 3, \ldots, n \}$, if it is not permitted to select two (or more) consecutive numbers?

Q1559 The diagram shows a circle with two inscribed semicircles; the semicircles are tangent to each other, and their diameters are parallel.



Show that the combined area of the semicircles is half the area of the circumscribed circle.

Q1560 Consider all positive integers up to 2018 which are a power of 2 times a power of 5. These numbers can be arranged into sets of three numbers, each set consisting of a geometric progression, with one number left over. What are the possibilities for the leftover number?