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The Radii of Hyper Circumsphere and Insphere through Equidistant Points

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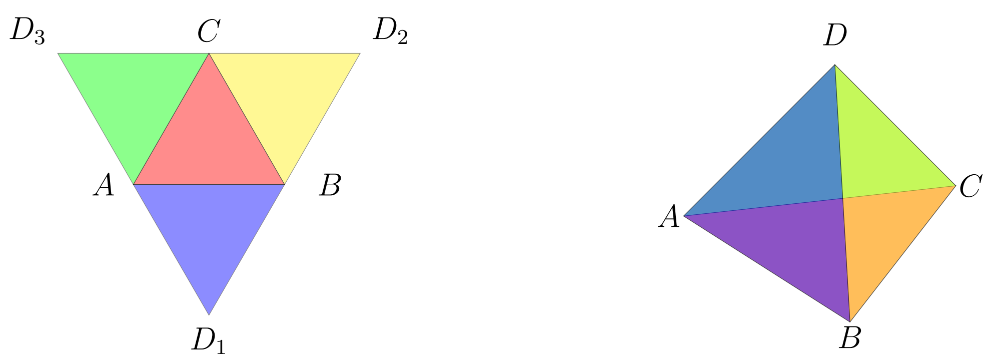
Three points A, B, and C of equal distance from each other form an equilateral triangle in $\mathbb{R}^2$. The reader can verify that it is not possible to construct a figure with 4 equidistant points in $\mathbb{R}^2$, as the points form a rhombus where the long diagonal is of different length to the other 5 edges. To form a figure with 4 equidistant points $A, B, C, D \in \mathbb{R}^3$, we extend an equilateral triangle from each of the sides AB, BC, and CA, and join the points D_1, D_2, and D_3 as in Figure 1(a). The result is a regular tetrahedron.



(a) Tetrahedron net

(b) Tetrahedron

Figure 1: Tetrahedron

Next, we extend from each of the faces of a regular tetrahedron $ABCD$ four new tetrahedra, as shown in Figure 2(a). By joining the points E_1, E_2, E_3, E_4 in $\mathbb{R}^4$, a figure with 5 equidistant points A, B, C, D, E is obtained, as shown in Figure 2(b). The *regular* 5-*simplex* formed in this way has 10 edges, 10 faces (each of which is a regular triangle) and 5 tetrahedra.

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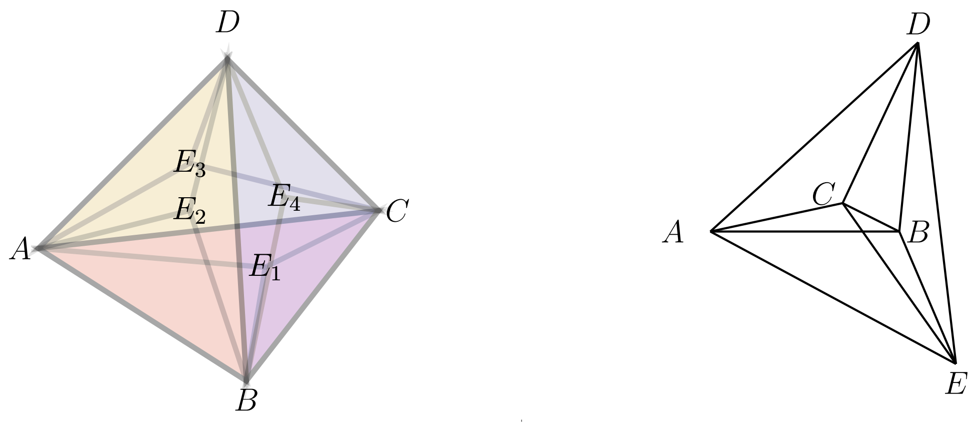

(a) extending from a tetrahedron (b) 5-simplex

Figure 2: Constructing a 5-simplex from a tetrahedron

In general, $n+1$ equidistant points $P_1, \ldots, P_{n+1}$ in $\mathbb{R}^n$ form a *regular* $(n+1)$-*simplex* S. It is formed by extending from each $(n-2)$-dimensional face of a regular n-simplex in $\mathbb{R}^{n-1}$ a regular n-simplex and joining their n new points into one new point.

The *circumsphere* is the hypersphere that goes through all of the $n+1$ points of S. The *insphere* is the largest hypersphere enclosed by these $n+1$ points.

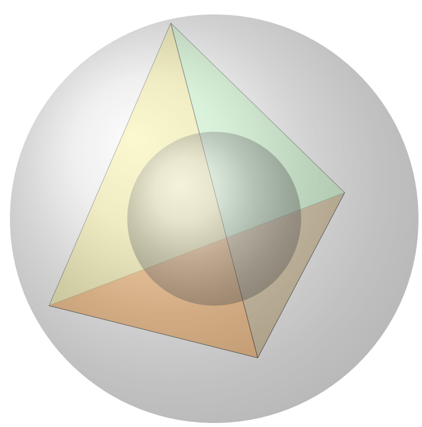

Figure 3: Insphere and circumsphere of a tetrahedron

The purpose of this paper is to derive formulae, given by the Theorem 1 to follow, for the radius R_n of the circumsphere and the radius r_n of the insphere for regular $(n+1)$-simplices in $\mathbb{R}^n$.

Example. We will now demonstrate the technique in determining the radius of the circumsphere and insphere of the regular tetrahedron in $\mathbb{R}^3$, as shown in Figure 3, by extending from an equilateral triangle.

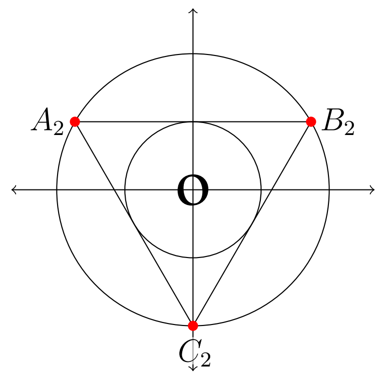

Figure 4: The circumsphere and insphere of a triangle

The points $A_2(-\frac{1}{2}, \frac{1}{\sqrt{12}})$, $B_2(\frac{1}{2}, \frac{1}{\sqrt{12}})$, $C_2(0, \frac{-1}{\sqrt{3}})$ in Figure 4 form an equilateral triangle with sides of length 1 satisfying the equation

$$x^2 + y^2 = \frac{1}{3} = \left(\frac{1}{\sqrt{3}}\right)^2 .$$

Therefore, the circumsphere of the triangle has radius $R_2 = \frac{1}{\sqrt{3}}$. Also, then the radius of the insphere of the triangle is $r_2 = \frac{1}{\sqrt{12}}$ since it is the perpendicular distance from the origin O to the line segment A_2B_2.

Now map the points A_2, B_2, C_2 to $A_3, B_3, C_3 \in \mathbb{R}^3$ by appending to each point a 0 as z-coordinate. Any point $P(0,0,z)$ is equidistant to A_3, B_3, C_3, so, to form a regular tetrahedron, choose z such that

$$z^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = 1 .$$

We see that $z = \frac{\sqrt{6}}{3}$, so the points

$$A_3\left(-\frac{1}{2}, \frac{1}{\sqrt{12}}, 0\right), \quad B_3\left(-\frac{1}{2}, \frac{1}{\sqrt{12}}, 0\right), \quad C_3\left(0, \frac{-\sqrt{3}}{3}, 0\right), \quad D_3\left(0, 0, \frac{\sqrt{6}}{3}\right)$$

form a regular tetrahedron with edges of length 1 in $\mathbb{R}^3$.

The next step is to shift the points to a sphere with center at the origin. Let P be the point $(0,0,d)$ and note that $PA_3 = PB_3 = PC_3$. We will now find P by considering the equation implied by $PC_3 = PD_3$:

$$\left(\frac{\sqrt{3}}{3}\right)^2 + d^2 = \left(d - \frac{\sqrt{6}}{3}\right)^2 .$$

Solving this equation yields $d = \frac{1}{\sqrt{24}}$, so P is the point $(0, 0, \frac{1}{\sqrt{24}})$. Since the point P lies on the z-axis, the distance d is the perpendicular distance from P to $\triangle ABC$ (which lies in the $x-y$ plane). By symmetry, it is also the perpendicular distance from P to $\triangle BCD$, $\triangle CDA$, and $\triangle DAB$, and is therefore the insphere radius of the regular tetrahedron, namely

$$r_3 = \frac{1}{\sqrt{24}}\,.$$

By shifting each point down by $\frac{1}{\sqrt{24}}$ in the z coordinate, the points

$$A_3'\left(\frac{-1}{2}, \frac{\sqrt{3}}{6}, \frac{-1}{\sqrt{24}}\right), \quad B_3'\left(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{-1}{\sqrt{24}}\right), \quad C_3'\left(0, \frac{-\sqrt{3}}{3}, \frac{-1}{\sqrt{24}}\right), \quad D_3'\left(0, 0, \frac{\sqrt{6}}{4}\right)$$

lie on the sphere

$$x^2 + y^2 + z^2 = \frac{3}{8} = \left(\sqrt{\frac{3}{8}}\right)^2.$$

These points form a regular tetrahedron centered around the origin. We see that the radius of circumsphere containing these points is

$$R_3 = \sqrt{\frac{3}{8}}\,.$$

Theorem 1. *Let $\{P_1, \ldots, P_{n+1}\}$ be $n+1$ points in $\mathbb{R}^n$ with constant distance 1 between each two points. Then the radii R_n and r_n of the circumsphere and insphere of the n-simplex $\{P_1, \ldots, P_{n+1}\}$ are, respectively,*

$$R_n = \sqrt{\frac{n}{2(n+1)}} \tag{1}$$

$$r_n = \frac{1}{\sqrt{2n(n+1)}}\,. \tag{2}$$

For instance,

$$R_2 = \frac{1}{\sqrt{3}}\,, \qquad r_2 = \frac{1}{\sqrt{12}}\,, \qquad R_3 = \sqrt{\frac{3}{8}}\,, \qquad r_2 = \frac{1}{\sqrt{24}}\,,$$

as we have seen in the example above.

Proof. The proof is by induction on n. We have proven the cases $n = 2, 3$, so assume that the theorem is true for $n \geq 3$. Consider $n+1$ points $\{P_1, \ldots, P_{n+1}\}$ in $\mathbb{R}^n$ with constant distance 1 between each two points. By shifting and rotating these points, we can let the centre of the circumsphere and insphere of these points be the origin so that the point $P_{n+1} = Z(n, -R_n)$, where

$$Z(n, z) = (\overbrace{0, 0, \ldots, 0}^{n-1}, z)\,.$$

Let the coordinates of each point be written as $P_k = (x_{k,1}, \dots, x_{k,n})$. Then

$$\sum_{i=1}^{n} x_{k,i}^2 = R_n^2 \,. \tag{3}$$

Map the points $P_1, \dots, P_{n+1} \in \mathbb{R}^n$ to the points $Q_1, \dots, Q_{n+1} \in \mathbb{R}^{n+1}$ by appending a 0 in coordinate position $n+1$.

Define

$$Q_{n+2} = Z(n+1, \sqrt{1-R_n^2}) \,.$$

Then, for each $k = 1, \dots, n+1$,

$$|Q_{n+2}Q_k|^2 = R_n^2 + 1 - R_n^2 = 1 \,.$$

Therefore, the points $Q_1, \dots, Q_{n+1}$ are each at distance 1 from Q_{n+2}, so these $n+2$ points form a regular $(n+1)$-simplex in $\mathbb{R}^{n+1}$.

Let $C = Z(n+1, d)$ for some d. For each $k = 1, \dots, n+1$, identity (3) implies that

$$|CQ_k|^2 = R_n^2 + d \,, \tag{4}$$

so C is equidistant to the points $Q_1, \dots, Q_{n+1}$.

We wish to determine d so that C also has this distance to Q_{n+2}. Since

$$R_n^2 + d^2 = \left(d - \sqrt{1-R_n^2}\right)^2,$$

we see that

$$d = \frac{1-2R_n^2}{2\sqrt{1-R_n^2}} \,. \tag{5}$$

Here, d is the perpendicular distance from the centre of the circle at C to the hyperplane containing $\{Q_1, \dots, Q_{n+2}\}$. Because of the symmetry of a regular $(n+1)$-simplex, d is also, for each $k = 1, \dots, n+2$, the projection of each point to the hyperplane through $\{Q_1, \dots, Q_{n+2}\} \backslash Q_k$. Therefore, C is the insphere to the points $Q_1, \dots, Q_{n+2}$, and so $r_{n+1} = d$. By (4),

$$r_{n+1} = \frac{1 - \frac{2(n-1)}{2n+2}}{2\sqrt{1 - \frac{n}{2n+2}}} = \frac{1}{\sqrt{2(n+1)(n+2)}} \,.$$

Since $Z(n+1, r_{n+1})$ is the centre of the circle through the points $Q_1, \dots, Q_{n+2}$, the radius of the circumsphere is given by distance between $Z(n+1, r_{n+1})$ and $Z\big(n+1, \sqrt{1-R_n^2}\big)$. Therefore, by induction assumption and (3),

$$\begin{aligned} R_{n+1} &= \sqrt{1-R_n^2} - r_{n+1} \\ &= \sqrt{1-R_n^2} - \frac{1-2R_n^2}{2\sqrt{1-R_n^2}} \\ &= \sqrt{1-R_n^2} \left(1 - \frac{1-R_n^2}{2(1-R_n^2)}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\sqrt{1-R_n^2}} \\
&= \frac{1}{2\sqrt{1-\frac{n}{2n+2}}} \\
&= \sqrt{\frac{n+1}{2(n+2)}}\,.
\end{aligned}
$$

Finally, map the points $Q_1, \ldots, Q_{n+2}$ to $R_1, \ldots, R_{n+2}$ by preserving the first n coordinates and subtracting d (calculated in (6)) from the $(n+1)^{th}$ coordinate. The points $R_1, \ldots, R_{n+2}$ are pairwise equidistant from each other, and therefore form the regular n-simplex in $\mathbb{R}^{n+1}$.

By induction, the proof is now complete. □

Theorem 1 provides a number of insights into the properties of a regular tetrahedron with $n+1$ vertices in the propositions below.

Proposition 2. *By the construction, the centres of the insphere and circumsphere coincide.*

Proposition 3. $R_n = nr_n$

Proposition 4. $\lim_{n\to\infty} R_n = \frac{1}{\sqrt{2}}$

Proposition 5. *As $n \to \infty$, each point pair P_iP_j subtend a right angle at the center C (see Figure 5).*

Proof. By the theorem above, we see that $R_n \to \frac{1}{\sqrt{2}}$ as $n \to \infty$. Therefore, Figure 5 and the cosine identity

$$\cos\theta = \frac{R_n^2 + R_n^2 - a^2}{2R_n^2}$$

imply that $\theta \to 90^\circ$ as $n \to \infty$. □

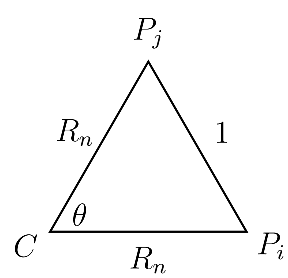

Figure 5: Angle subtended by an edge to the centre of circumsphere

Michelle, thank you for the song that summer sings.