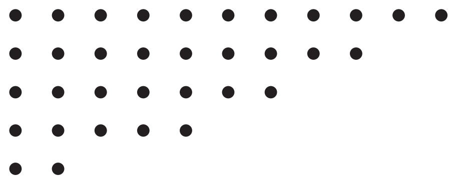
Parabola Volume 54, Issue 2 (2018)

Solutions 1551–1560

Q1551 We have a pattern of 34 dots arranged as shown.



It is permitted to remove any three dots, provided that one of them is exactly midway between the other two (the three dots may form a line in any direction – horizontal, vertical, diagonal or oblique); then to remove another three dots under the same condition; and so on. If we remove 33 dots, which are the possibilities for the remaining dot?

SOLUTION Replace the dots by numerical labels as shown.

$$\begin{array}{ccccccccccc}
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & & \\
2 & 0 & 1 & 2 & 0 & 1 & 2 & & & & \\
0 & 1 & 2 & 0 & 1 & & & & & & \\
1 & 2 & & & & & & & & &
\end{array}$$

In this pattern, any three dots which may be removed together (that is, one of them is midway between the other two) either will have the labels $0, 1, 2$, or will have the same label three times. In each case, the sum of the labels removed will be a multiple of 3. Now the total of all the labels is

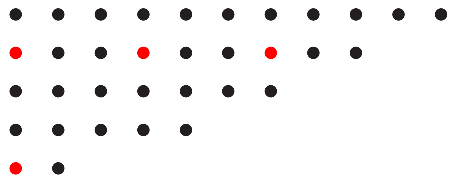
$$(11 \times 0) + (12 \times 1) + (11 \times 2) = 34\ ;$$

if we keep on decreasing this total by multiples of 3 until only one label remains, then it must be 1. Now label the dots in a similar but different way.

$$\begin{array}{ccccccccccc}
0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \\
2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & & \\
1 & 2 & 0 & 1 & 2 & 0 & 1 & & & & \\
0 & 1 & 2 & 0 & 1 & & & & & & \\
2 & 0 & & & & & & & & &
\end{array}$$

The total of these labels is 32; as above, whenever we remove three permitted dots, the total must diminish by a multiple of 3; so the remaining label at the end must be 2.

Therefore, the dots which could remain are those which have the label 1 in the first diagram and 2 in the second: these are the first, fourth and seventh dots in the second row, and the first dot in the fifth row.



To confirm that it is possible to leave a dot in the place marked by the red dot in the fifth row, we can remove the groups of three dots marked $\mathrm{A}, \mathrm{B}, \ldots, \mathrm{K}$ in the following diagram; to confirm that each of the other red dots can also also be the survivor is left for you as an exercise.

$$\begin{array}{ccccccccccc}
\mathrm{A} & \mathrm{A} & \mathrm{A} & \mathrm{B} & \mathrm{B} & \mathrm{B} & \mathrm{C} & \mathrm{C} & \mathrm{C} & \mathrm{J} & \mathrm{K} \\
\mathrm{D} & \mathrm{F} & \mathrm{F} & \mathrm{F} & \mathrm{G} & \mathrm{G} & \mathrm{G} & \mathrm{J} & \mathrm{K} & & \\
\mathrm{D} & \mathrm{E} & \mathrm{H} & \mathrm{H} & \mathrm{H} & \mathrm{J} & \mathrm{K} & & & & \\
\mathrm{D} & \mathrm{E} & \mathrm{I} & \mathrm{I} & \mathrm{I} & & & & & & \\
\bullet & \mathrm{E} & & & & & & & & &
\end{array}$$

Q1552 Pat and Sandy are sharing a project. Pat works for 12 days and completes more than half the job; Sandy then takes over and finishes the job, taking another 12 days. If, instead, Pat had done exactly half the job before Sandy took over for the rest, and if they both worked at the same rate as in the previous scenario, then the whole project would have taken 25 days. How many days would Pat have worked for?

SOLUTION Suppose that in the second scenario Pat works for x days and Sandy for $25 - x$. Then to complete the whole project, Pat would have taken $2x$ days and Sandy $2(25 - x)$. The proportions of the job done by Pat and by Sandy in 12 days each would have been

$$\frac{12}{2x} \quad \text{and} \quad \frac{12}{2(25-x)}$$

respectively. Since by both doing this the job was completed, these fractions must add up to 1, and we then easily solve a quadratic equation:

$$\begin{aligned}
\frac{12}{2x} + \frac{12}{2(25-x)} = 1 \quad &\Leftrightarrow \quad 6(25-x) + 6x = x(25-x) \\
&\Leftrightarrow \quad x^2 - 25x + 150 = 0 \\
&\Leftrightarrow \quad x = 10 \text{ or } x = 15\,.
\end{aligned}$$

It is clear from the second sentence in the question that Pat takes less than 12 days to do half the job, and therefore we reject $x = 15$. So, in the second scenario, Pat works for 10 days (and Sandy for 15 days).

Q1553 For every positive integer n we define a positive integer $f(n)$ such that the following property is true:

$$\text{if } m, n \text{ are any positive integers, then } m^2 + f(n) \mid mf(m) + n,$$

where the notation $a \mid b$ denotes that a is a factor of b. It is easy to see that $f(n) = n$ for all n is one possibility; prove that there is no other possibility.

SOLUTION First take $m = f(1)$ and $n = 1$. Then the condition says that

$$f(1)^2 + f(1) \mid f(1)f(f(1)) + 1 \;;$$

clearly $f(1)$ is a factor of the left hand side, so $f(1)$ is a factor of the right hand side, so $f(1)$ is a factor of 1. Therefore $f(1) = 1$.

Now take $m = 1$ in the given condition. For any n we have

$$\begin{aligned} 1 + f(n) \mid f(1) + n \quad &\Rightarrow \quad 1 + f(n) \mid 1 + n \\ &\Rightarrow \quad 1 + f(n) \le 1 + n \\ &\Rightarrow \quad f(n) \le n \,. \end{aligned}$$

Finally, in the given condition replace n by 1 and then m by n. Then

$$\begin{aligned} n^2 + f(1) \mid nf(n) + 1 \quad &\Rightarrow \quad n^2 + 1 \mid nf(n) + 1 \\ &\Rightarrow \quad n^2 + 1 \le nf(n) + 1 \\ &\Rightarrow \quad n^2 \le nf(n) \\ &\Rightarrow \quad f(n) \ge n \,. \end{aligned}$$

Since we have shown that $f(n) \le n$ and $f(n) \ge n$, we have $f(n) = n$ for all n.

Q1554 Suppose it is possible to arrange the numbers $1, 2, 3, \ldots, mn$ in an array of m rows and n columns in such a way that each of the rows has the same sum, and each of the columns has the same sum. (Compare problem 1545, solution in the previous issue). Show that it is possible to arrange the numbers $1, 2, 3, \ldots, m^2n$ in an array of m rows and mn columns with the same property.

SOLUTION First we shall illustrate with an example; then we shall prove that a similar procedure always works. Take the case $m = 3$, $n = 5$; the numbers $1, 2, 3, \ldots, 15$ can be arranged in a 3 by 5 array

15	6	2	7	10
1	14	9	5	11
8	4	13	12	3

in which, as you can easily check, each row adds up to 40 and each column adds up to 24. Since $m = 3$ we shall write this array 3 times side by side; and we shall colour the numbers to assist the argument:

15	6	2	7	10	15	6	2	7	10	15	6	2	7	10
1	14	9	5	11	1	14	9	5	11	1	14	9	5	11
8	4	13	12	3	8	4	13	12	3	8	4	13	12	3

Now increase all the green numbers by 15 and all the blue numbers by 30 to obtain

15	6	2	7	10	45	36	32	37	40	30	21	17	22	25
16	29	24	20	26	1	14	9	5	11	31	44	39	35	41
38	34	43	42	33	23	19	28	27	18	8	4	13	12	3

In this array every row contains 5 red, 5 blue and 5 green numbers and therefore has been increased by 5×15 and 5×30 for a total of 265; similarly, each column now has a total of $24+15+30 = 69$. Moreover, the numbers in the array are $1, 2, \ldots, 15$, unaltered; and $1, 2, \ldots, 15$ increased by 15 to give $16, 17, \ldots, 30$; and $1, 2, \ldots, 15$ increased by 30 to give $31, 32, \ldots, 45$: that is, the array contains the numbers $1, 2, \ldots, 45$, and is a 3 by 15 array of the type we wanted.

To show that this is always possible we proceed as follows. Given a suitable m by n array, write it down m times side by side so that we have an m by mn array consisting of m subarrays, each of size m by n. Now

- in the first subarray, increase every number in the second row by mn, every number in the third row by $2mn, \ldots$ and every number in the mth row by $(m-1)mn$;
- in the second subarray, increase every number in the third row by mn, every number in the fourth row by $2mn, \ldots$, every number in the mth row by $(m-2)mn$ and every number in the first row by $(m-1)mn$;
- and so on.

Then we have an m by mn array with the following properties.

- The second row of the first subarray, the third row of the second subarray, $\ldots$, the mth row of the $(m-1)$th subarray and the first row of the mth subarray contained, when they were created, the numbers from 1 to mn; then they were all increased by mn; so they now contain the numbers $mn+1$ to $2mn$. Similarly, the rows following these now contain the numbers $2mn+1$ to $3mn$, and so on: the whole array contains the numbers from 1 to m^2n.
- The sum of the entries was the same in every column; now every column has had one entry increased by mn, one by $2mn, \ldots$ and one by $(m-1)mn$; the increases are the same in every case, so the sum of the entries still is the same in every column.
- Each row contained n elements repeated m times; n of these elements were then increased by mn, and n of them by $2mn$, and so on. So the sum of elements in a row is now

  $$\langle \text{original row sum} \rangle + n(mn) + n(2mn) + \cdots + n((m-1)mn) ,$$

  and this is the same for every row.

Thus, the new array is an m by mn array of the type we wanted.

Q1555 Show that if the quadratic equation

$$x^2 - px + p = 0$$

has two (real) solutions x_1 and x_2, then $x_1^2 + x_2^2 > 2(x_1 + x_2)$.

SOLUTION From the well–known relations between roots and coefficients of a quadratic, we have

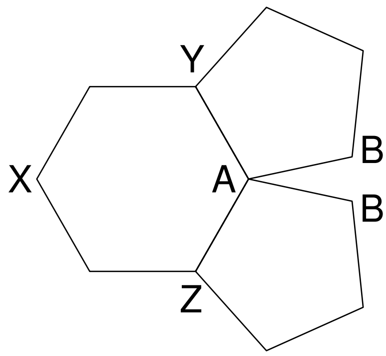
$$x_1 + x_2 = p \quad \text{and} \quad x_1 x_2 = p \, .$$

Therefore

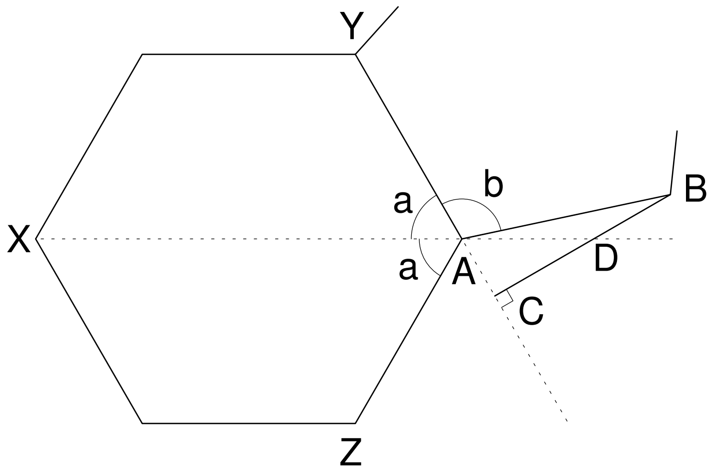
$$\begin{aligned} x_1^2 + x_2^2 - 2(x_1 + x_2) &= (x_1 + x_2)^2 - 2x_1x_2 - 2(x_1 + x_2) \\ &= p^2 - 4p \, . \end{aligned}$$

But $p^2 - 4p$ is the discriminant of this quadratic ($b^2 - 4ac$ with $a = 1$, $b = p$, $c = p$), and since the quadratic has two real roots, the discriminant is positive. Thus $x_1^2 + x_2^2 > 2(x_1 + x_2)$, as claimed.

Q1556 A shape consisting of a regular hexagon and two regular pentagons is cut out of cardboard; the pentagons are bent upwards along the lines AY and AZ until the two points marked B meet. What is then the angle $\angle XAB$?



SOLUTION We solve a more general case with $\angle XAY = \angle XAZ = a$ and $\angle BAY = b$. Draw a perpendicular from B to AY extended, meeting AY at C; extend XA to meet BC at D.



Since the pentagon $YAB\cdots$ is rotated around YC, the perpendicular line BC will always remain directly above its initial position. When the point B meets the other point

B (from the diagram in the question), they will be directly above the line XA; and also, as already shown, directly above BC; so they will be directly above the point D. We will denote the location of the points B when this happens by B'; so the problem is to find the angle $\angle XAB'$.

Since $\angle ADB'$ is a right angle, we have

$$\cos\angle DAB' = \frac{AD}{AB'} \ .$$

Now, triangles $\triangle ACB$ and $\triangle ACD$ are both right–angled, so

$$AC = AB\cos(\pi - b) \quad \text{and} \quad AC = AD\cos a \ ;$$

moreover, the intervals AB' and AB have the same length as one is a rotation of the other. So

$$\cos\angle DAB' = \frac{AD}{AB} = \frac{AC/AB}{AC/AD} = \frac{\cos(\pi - b)}{\cos a} \ ,$$

and the required angle is the supplement of this,

$$\angle XAB' = \pi - \arccos\Bigl(\frac{\cos(\pi - b)}{\cos a}\Bigr) = \arccos\Bigl(\frac{\cos b}{\cos a}\Bigr) \ .$$

In the specific case asked we have $a = 60^\circ$, $b = 108^\circ$ and so

$$\angle XAB' = \arccos\Bigl(\frac{\cos 108^\circ}{\cos 60^\circ}\Bigr) = 128.2^\circ \ .$$

Q1557 Eric is playing a game in which he rolls a (normal, six–sided) die three times, and wins if his three rolls are all different and in increasing order. For example, $1, 4, 5$ wins, but $4, 1, 5$ loses, and so does $4, 5, 5$. In the middle of the game Eric calls you on the phone and tells you that his second roll was bigger than his first. If the game continues, what is Eric's chance of winning?

SOLUTION Eric's second roll cannot have been 1, as it was bigger than his first. There is one way that the second roll could have been 2 (he threw $1, 2$); and he then has four winning possibilities on the third roll $(3, 4, 5, 6)$. There are two ways that the second roll could have been 3 (he threw $1, 3$ or $2, 3$); and in each of these two cases, he has three winning possibilities on the third roll $(4, 5, 6)$. Continuing to think in this way gives the following.

second roll	2	3	4	5	6
number of possibilities	1	2	3	4	5
number of winning third rolls	4	3	2	1	0

So, the number of possibilities for the first two rolls is $1 + 2 + 3 + 4 + 5 = 15$, and the number for all three is $15 \times 6 = 90$. The number of winning possibilities is

$$(1 \times 4) + (2 \times 3) + (3 \times 2) + (4 \times 1) = 20 \ ,$$

and so Eric's chance of winning is $\frac{20}{90} = \frac{2}{9}$.

Q1558 Let n and k be positive integers with $n \geq 2k - 1$. In how many ways can k numbers be selected from $\{1, 2, 3, \ldots, n\}$, if it is not permitted to select two (or more) consecutive numbers?

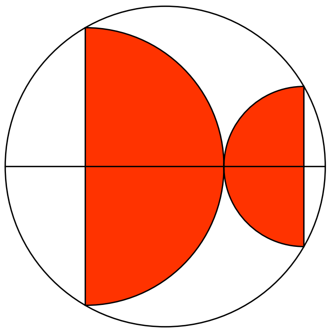
SOLUTION Consider a sequence of $n + 1$ letters Y and N, where the kth letter is Y if the number k is chosen, and N if not. This will give a permissible choice if and only if

- the $(n + 1)$th letter is N (because $n + 1$ may not be chosen);
- there are exactly k letters Y;
- two or more consecutive Ys never occur.

To achieve this we can "bind" each Y to a following N: thus we have to arrange k pairs YN and $n + 1 - 2k$ individual letters N; the number of ways of doing so is given by the binomial coefficient

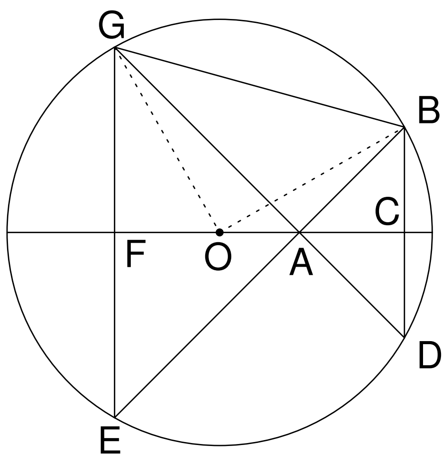
$$\binom{n+1-k}{k} = C(n+1-k, k) = {}^{n+1-k}C_k \,.$$

Q1559 The diagram shows a circle with two inscribed semicircles; the semicircles are tangent to each other, and their diameters are parallel.



Show that the combined area of the semicircles is half the area of the circumscribed circle.

SOLUTION The area of a circle is equal to $\pi/2$ times the area of the inscribed square. So we consider the following diagram.



Triangles $\triangle ABD$ and $\triangle AEG$ are right–angled isosceles triangles, and the total area of the two semicircles is

$$\frac{\pi}{2}(\text{area of }\triangle ABD + \text{area of }\triangle AEG) = \frac{\pi}{4}(AB^2 + AG^2)\ .$$

Angles $\angle BAC$ and $\angle GAF$ are both $45°$, so $\angle BAG$ is a right angle and by Pythagoras $AB^2 + AG^2 = BG^2$. Furthermore $\angle BEG = 45°$; the angle at the centre of the circle subtended by the same chord is twice this, $\angle BOG = 90°$; and both OB and OG are radii of the circles, so $BG = \sqrt{2}\,OB$. Putting all this information together, the total area of the two semicircles is

$$\frac{\pi}{4}BG^2 = \frac{\pi}{4}(2\,OB^2) = \frac{1}{2}\,\pi\,OB^2$$

which is half the area of the circumscribed circle.

NOW TRY Problem 1562.

Q1560 Consider all positive integers up to 2018 which are a power of 2 times a power of 5. These numbers can be arranged into sets of three numbers, each set consisting of a geometric progression, with one number left over. What are the possibilities for the leftover number?

SOLUTION Because of the limit of 2018 on their size, the numbers we have are

$$\begin{aligned}
&2^0 5^0 = 1,\ 2^1 5^0 = 2,\ 2^2 5^0 = 4,\ 2^3 5^0 = 8, \ldots,\ 2^{10} 5^0 = 1024,\\
&2^0 5^1 = 5,\ 2^1 5^1 = 10,\ 2^2 5^1 = 20, \ldots,\ 2^8 5^1 = 1280,\\
&2^0 5^2 = 25,\ 2^1 5^2 = 50, \ldots,\ 2^6 5^2 = 1600,\\
&2^0 5^3 = 125, \ldots,\ 2^4 5^3 = 2000,\\
&2^0 5^4 = 625,\ 2^1 5^4 = 1250\ .
\end{aligned}$$

Let a be the sum of all the exponents on powers of 2 among these numbers, and b the sum of all the exponents on powers of 5. We calculate

$$\begin{aligned}
a &= (5\times 0) + (5\times 1) + (4\times 2) + (4\times 3) + (4\times 4)\\
&\qquad + (3\times 5) + (3\times 6) + (2\times 7) + (2\times 8) + 9 + 10\\
&= 123\\
b &= (11\times 0) + (9\times 1) + (7\times 2) + (5\times 3) + (2\times 4)\\
&= 46\ .
\end{aligned}$$

Now suppose that we have a set of three numbers in geometric progression, in which the middle term is $2^p 5^q$ and the common ratio is $2^s 5^t$. The numbers in the progression are

$$2^{p-s}5^{q-t}\ ,\quad 2^p 5^q\ ,\quad 2^{p+s}5^{q+t}\ ;$$

the sum of the exponents on powers of 2 is $3p$, and on powers of 5 is $3q$. So if we place these three numbers in a set and from now on ignore them, the sum of the exponents on powers of 2 has decreased by a multiple of 3; and likewise for the sum of the exponents on powers of 5. Since 123 has remainder 0 when divided by 3 and 46 has remainder 1 and these remainders will never change, our leftover term $2^x 5^y$ must have an x value with remainder 0 and a y value with remainder 1 when divided by 3. Within the set of numbers we began with, the possibilities are

$$x = 0\,,\ y = 1 \ \text{ or } \ x = 0\,,\ y = 4 \ \text{ or } \ x = 3\,,\ y = 1 \ \text{ or } \ x = 6\,,\ y = 1$$

and so the remaining number is

$$5 \quad \text{or} \quad 625 \quad \text{or} \quad 40 \quad \text{or} \quad 320\,.$$

NOW TRY Problem 1561, in which you are asked to give **a different way** of solving this problem.