Parabola Volume 54, Issue 3 (2018)

Factorial Countdown: Making numbers out of smaller ones

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1 Introduction

Those of you who do not live in Britain may not be familiar with *Countdown*. For the uninitiated, this is a game show with a round in which the players are given a set of numbers and are asked to use some elementary operations to reach a numeric target. All numbers used, including intermediate results of calculations, must be positive integers ($\mathbb{Z}^+$). In this paper, I consider a different, in some ways stricter, version of the game, called *Factorial Countdown*: you must use every positive integer less than the target; you may only use each once; and you are given a very restricted subset of operations. The question is whether it is possible to reach the target; this turns out to depend on the factorisation of the target in a strange series of ways. Note that this paper will not use the word "game" in the standard mathematical sense of the term. Game theorists, you may look away now.

It is useful to know that $1 + 2 + 3 + 4 + 5 + \cdots + n = \frac{n(n+1)}{2}$. There are many proofs, including a couple in the Parabola article here.

2 Defining the game

We are given a target, which is some positive integer t. Much as in Countdown, we must try to construct an expression out of other numbers which evaluates to t; unlike in Countdown, we must use all positive integers less than t. For example, if $t = 5$, then we would have to use 1, 2, 3, and 4 once and only once each. The complexity of the game arises from the choice of allowed operations. We select from addition, subtraction, multiplication, and division, denoted A, S, M, and D, respectively. We name games based on the operations allowed. Different choices give very different games – for example, as will be shown later, the MD game, in which we are allowed only to multiply and divide, cannot be won, while the SM game, in which we are only allowed to multiply and subtract, can always be won if $t > 3$.

Since there are four operators, and each can either be included or excluded from any game, and there must always be at least one operator in each game, we have $2^4 - 1 = 15$ different possible games. Simple enumeration shows that they are A, S, M, D, AM, AS, AD, SM, SD, MD, ASM, ASD, AMD, SMD, and ASMD.

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We are allowed unlimited parentheses. At no point may we use negative numbers or non-integers, so $\frac{2}{1\div 2} = 4$ would be invalid, for instance. It is interesting to see first that no game can possibly be won for $t = 2$, as this gives us only the number 1 to work with and all possible games use only binary operators.[2]

3 A, S, M, and D: Single-operator games

We see that $1+2+3+4+5+\cdots+n = \frac{n(n+1)}{2}$ and $1\times 2\times 3\times\cdots\times n = n!$ both grow faster than n does, so, for t greater than a certain threshold, the A and M games, which consist of calculating $\frac{(t-1)(t)}{2}$ and $(t-1)!$ respectively, cannot be won. Calculation demonstrates that this threshold is 1 for the M game and 3 for the A game. Furthermore, we see that the maximum value attainable during the S game is

$$(t-1) - [1+2+3+4+\cdots+(t-2)] = t - 1 - \frac{(t-2)(t-1)}{2}\,.$$

Again, this becomes negative after a certain threshold, which calculation shows to be 4, as $3-2-1 = 0$ but $4-3-2-1 = -2$. Brute-force calculations for $t = 1, 2, 3$ demonstrate that the S game cannot actually be won for any of them either. Meanwhile, the D game is incapable of generating numbers larger than $t - 1$, so cannot be won. To conclude, the A game can be won for $t = 3$, while the S, M, and D games cannot be won at all.

4 AM and SD: strictly increasing or decreasing games

The AM game cannot be won unless $t = 3$, in which case it reduces to the A game. This is because addition and multiplication are both strictly increasing – add anything to m and you get, at the very least, $m + 1$. When playing the AM game for $t > 3$, we will have to apply such strictly increasing operations to $t - 1$ at least twice, leading to an end result that is at least $t + 1$. Thus, we cannot reach t. Similarly, the SD game cannot be won.

5 The MD game

The MD game consists of calculating $f_1 \times f_2 \times f_3 \times \cdots \times f_{t-1}$, where each f_k could be either k or $\frac{1}{k}$. By rewriting the second option as $\frac{k}{k^2}$, it becomes clear that this game is equivalent to solving $\frac{(t-1)!}{t_1^2 t_2^2 \cdots t_n^2} = t$ for $t_k < t$. This rearranges to $(t-1)! = t \times t_1^2 t_2^2 \cdots t_n^2$. Now, aside from factors of t, all prime factors on the right-hand side have even powers. Calculation proves this to be impossible for $t \leq 3$. However, for $t > 3$, by Bertrand's Postulate, there is a prime number between $\frac{t}{2}$ and $t-1$. This prime appears exactly once in the factorisation of $(t-1)!$, and never in the factorisation of t. This is a contradiction, so the MD game cannot be won.

[2]We say that an operator is binary when it requires two inputs. Addition is binary.

6 The AS game

The AS game can be won for exactly half of all positive integers. We wish to solve the equation

$$(-1)^{a_1}(1) + (-1)^{a_2}(2) + (-1)^{a_3}(3) + \cdots + (-1)^{a_{t-1}}(t-1) = t\,,$$

where $a_k \in 0, 1$. This is equivalent to

$$(1 + 2 + 3 + \cdots + (t-1)) - 2(1a_1 + 2a_2 + 3a_3 + \cdots + (t-1)a_{t-1}) = t\,.$$

If we take remainders when dividing by 2, the second term vanishes and, remembering our formula for the sum of the first n integers, we have $\frac{(t-1)(t)}{2} \equiv t \pmod{2}$. If $t = 4k$ or $t = 4k+3$, then either both sides are even or both are odd and the congruence is satisfied. If, however, $t = 4k+1$ or $t = 4k+2$, then one side is even while the other is odd, which is a contradiction. Therefore, the AS game can only be won for $t = 4k$ or $t = 4k+3$.

When $t = 4k$, we have $4k-1$ numbers to work with. These we add and subtract in groups of four:

$$(t-1)+(t-2)-(t-3)-(t-4)]+[(t-5)+(t-6)-(t-7)-(t-8)]+\cdots+[7+6-5-4]+[3+2-1]\,.$$

The last group only has three numbers in it. Each of these groups sums to 4, and we have $\frac{t}{4}$ such groups, so the sum is equal to t and the game is won.

When $t = 4k+3$, we group the $4k+2$ given numbers similarly:

$$[(t-1)+(t-2)-(t-3)-(t-4)]+[(t-5)+(t-6)-(t-7)-(t-8)]+\cdots+[6+5-4-3]+[2+1]\,.$$

The first k groups all sum to 4, and the final group sums to 3, giving us $4k+3 = t$ to win the game.

7 The SM game

The SM game can always be won for sufficiently large t. Again, our strategy depends on the remainder of t when divided by 4. One strategy that we will often use, which I call *multiplying doubles away*, consists of taking two consecutive numbers, subtracting the smaller from the larger to get 1, and multiplying by that 1. For example, to get 4 from $4, 3, 2$, we might say $4(3-2) = 4$.

If t is odd and $t - 7 > 5$, then we begin by forming the difference of two quadratics with the largest four numbers available:

$$(t-1)(t-2) - (t-3)(t-4) = 4t - 10\,.$$

We then subtract the next three to get

$$4t - 10 - (t-5) - (t-6) - (t-7) = t + 8\,.$$

We now subtract $2, 3, 4, 5$ in a certain order to get $t + 8 - 2 - (5 - 3) - 4$, multiply by 1, and simplify to get t. Since t is odd, we are given an even number of numbers less than t. We have used $4 + 3 + 5 = 12$ of these numbers, leaving us with an even number left over. Since we have only taken the largest and smallest possible numbers, we may pair up the rest and multiply doubles away.

For example, take $t = 17$. Then we have

$$[16 \times 15 - 14 \times 13 - 12 - 11 - 10 - 2 - (5 - 3) - 4] \times 1 \times (9 - 8) \times (7 - 6) = 17 = t\,.$$

If $t < 12$, then we obviously cannot do this. Evaluating all possibilities quickly shows that $t = 1, 3$ cannot be won. In contrast, solutions for $t = 5$ and $t = 7$ exist: for instance, $(4 - 2) \times 3 - 1 = 5$ and $6 \times 5 - (4 \times 3 \times 2 - 1) = 7$. Solutions for $t = 9$ and $t = 11$ exist as well, and are left as an exercise for the reader.

If t is even, then our solution involves even more multiplying away of doubles. If $t = 4k + 2$, then we multiply doubles away to get rid of all numbers except for $\frac{t}{2}$ and 2. For example,

$$10 = (9 - 8) \times (7 - 6) \times 5 \times (4 - 3) \times 2 \times 1\,.$$

If, on the other hand, $t = 4k$, then we get rid of all numbers except for $\frac{t}{2}$, $\frac{t}{2} - 1$, and $\frac{t}{2} + 1$, which give

$$\frac{t}{2} \times \left[\left(\frac{t}{2} + 1\right) - \left(\frac{t}{2} - 1\right)\right] = 2 \times \frac{t}{2} = t\,.$$

For example,

$$12 = (11 - 10) \times (9 - 8) \times (7 - 5) \times 6 \times (4 - 3) \times (2 - 1)\,.$$

This wins the game for all even t greater than 2.

8 The AD game

This game can be won for all $t > 4$, and also for $t = 3$ by reduction to the A game. When t is odd, we begin by calculating $\frac{1+2}{3} = 1$. We then iteratively add this 1 to the next smallest number and divide by the next to, once again, get 1. This continues until we can add $1 + (t - 1) = t$. For example, to get 7, we calculate $\frac{1+2}{3} + 4 = 5$, $\frac{5}{5} + 6 = 7$, giving us the resulting expression

$$(((1 + 2) \div 3) + 4) \div 5 + 6 = 7\,.$$

When t is even, we proceed in the same way but begin with $(((1 + 2 + 3) \div 6) + 4) \div 5$. Obviously, this is only viable for $t > 6$. We solve $t = 6$ by $((1+4)\div 5)+2+3$. Evaluating all possibilities shows that this cannot be won for $t = 4$.

9 Three-operator games

Since each three-operator game contains three two-operator games, we are already well on our way to winning them. The ASM game can be won for $t > 4$ by reduction to SM, and for $t = 3$ by reduction to A. The ASD game can be won for $t > 4$ by reduction to the AD game, for $t = 3$ by reduction to A, and for $t = 4$ by reduction to AS. The AMD and SMD games can be won in the same way, except that $t = 4$ cannot be won.

10 Conclusion

Since there is some three-operator game that can be won for any $t \geq 3$, the four-operator game can be won for $t \geq 3$. We tabulate our results briefly here:

Game	**Winnable?**
A	$t = 3$
S, M, D	No
AS	$t = 4k$, $t = 4k + 3$
AM, SD, MD	No
SM	$t > 3$
AD, AMD	$t = 3$, $t > 4$
SMD	$t > 3$
ASM, ASD, ASMD	$t \geq 3$

Addition and subtraction are often mentally "paired up", as are multiplication and division. After all, they are each others' inverses. It is notable that a winnable game for most integers requires members from both of these pairs, and that winning for all integers requires the entirety of one pair as well.

Finding a winning strategy for real Countdown is left as an exercise for the interested reader.