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A Mean Inequality and Applications Thereof

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The *arithmetic mean* and the *root mean square* of real numbers $x_1, \ldots, x_n$ are, resp.,

$$\frac{x_1 + \cdots + x_n}{n} \qquad \text{and} \qquad \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}\,.$$

The purpose of this note is to show that these means the first mean is always bound how these two means relate to each other, by way of the inequality in Theorem 1 below, and to show how this simple inequality can be useful for solving systems of equations.

Theorem 1.

$$\left|\frac{x_1 + \cdots + x_n}{n}\right| \le \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Proof. Since

$$0 \le \sum_{i,j=1}^{n} (x_i - x_j)^2 = \sum_{i,j=1}^{n} (x_i^2 + x_j^2 - 2x_i x_j) = 2n \sum_{i=1}^{n} x_i^2 - 2 \sum_{i,j=1}^{n} x_i x_j = 2n \sum_{i=1}^{n} x_i^2 - 2\Big(\sum_{i=1}^{n} x_i\Big)^2,$$

it follows that

$$\Big(\frac{1}{n} \sum_{i=1}^{n} x_i\Big)^2 \le \frac{1}{n} \sum_{i=1}^{n} x_i^2\,.$$

Furthermore, equality holds if and only if

$$0 = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i - x_j)^2$$

which in turn is true exactly when $x_i = x_j$ for all $i, j = 1, \ldots, n$. □

Theorem 1 is a special case of Chebyshev's Sum Inequality which states that

$$\Big(\frac{1}{n} \sum_{i=1}^{n} x_i\Big)\Big(\frac{1}{n} \sum_{i=1}^{n} y_i\Big) \le \frac{1}{n} \sum_{i=1}^{n} x_i y_j$$

whenever $x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \le y_2 \le \cdots \le y_n$; this more general inequality can be proved be modifying the above proof slightly (feel free to try to do this!).

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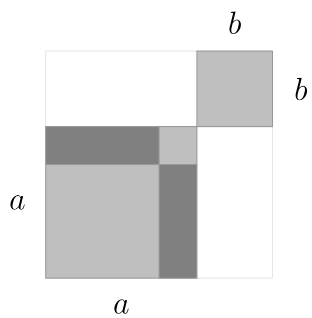
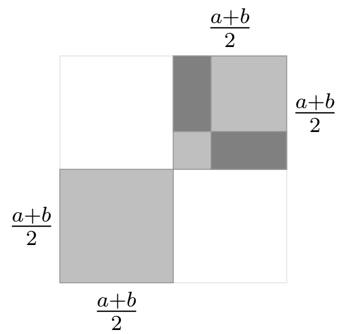
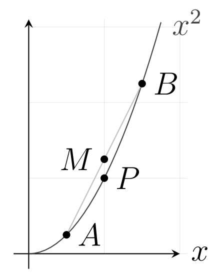
For the case in which there are just $n = 2$ variables, Theorem 1 is easily seen in at least two visual ways, below.







In the first two figures above, the grey parts have total area $a^2 + b^2$ and $2\big(\frac{a+b}{2}\big)$, respectively, where, without loss of generality, $a \geq b$. They share the same three light-grey squares but differ in their dark-grey rectangles; these are bigger in the first figure than in the second figure, so the total grey area of the first figure is bigger than the total grey area of the second:

$$a^2 + b^2 \geq 2\Big(\frac{a+b}{2}\Big)^2 ,$$

so

$$\Big|\frac{a+b}{2}\Big| \leq \sqrt{\frac{a^2+b^2}{2}} .$$

In the third figure, the two points A and B on the parabola $y = x^2$ have coordinates (a, a^2) and (b, b^2), say. The midpoint M between these two points has coordinates $\big(\frac{a+b}{2}, \frac{a^2+b^2}{2}\big)$ and lies vertically above a point P on the parabola. Since P has coordinates $\Big(\frac{a+b}{2}, \big(\frac{a+b}{2}\big)^2\Big)$, we again see that

$$\Big(\frac{a+b}{2}\Big)^2 \leq \frac{a^2+b^2}{2} .$$

Applications

We now present three examples to demonstrate how Theorem 1 can sometimes be used to solve systems of equations.

Example 1. *Let us find all real solutions to the following system of equations:*

$$a - 3b - 5c + 7d = 420 \qquad \textit{and} \qquad a^2 + 9b^2 + 25c^2 + 49d^2 = 44100 .$$

Set $x_1 = a$, $x_2 = -3b$, $x_3 = -5c$, $x_4 = 7d$, *and* $n = 4$. *Then*

$$\frac{x_1 + \cdots + x_n}{n} = \frac{a - 3b - 5c + 7d}{4} = \frac{420}{4} = 105$$

and

$$\sqrt{\frac{x_1^2+\cdots+x_n^2}{n}} = \sqrt{\frac{a^2+9b^2+25c^2+49d^2}{4}} = \sqrt{\frac{44100}{4}} = \sqrt{11025} = 105\,,$$

so

$$\frac{x_1+\cdots+x_n}{n} = \sqrt{\frac{x_1^2+\cdots+x_n^2}{n}}\,.$$

By Theorem 1, $x_1 = x_2 = \cdots = x_n = 105$; that is, $a = -3b = -5c = 7d = 105$. Hence,

$$a = 105\,, \qquad b = -35\,, \qquad c = -21\,, \qquad d = 15\,.$$

Example 2. *Let us find all real solutions to the following system of equations:*

$$a - b + 3c + 5d = -100 \qquad \textit{and} \qquad a^2 + b^2 + 26c^2 + 89d^2 = 1764\,.$$

Set $x_1 = a$, $x_2 = -b$, $x_3 = 3c$, $x_4 = 5d$, and $n = 4$. Then

$$\frac{|x_1+\cdots+x_n|}{n} = \frac{|a-b-3c+5d|}{4} = \frac{|-100|}{4} = 25$$

and

$$\sqrt{\frac{x_1^2+\cdots+x_n^2}{n}} = \sqrt{\frac{a^2+b^2+9c^2+25d^2}{4}} \leq \sqrt{\frac{a^2+b^2+26c^2+89d^2}{4}} = \sqrt{\frac{1764}{4}} = 21\,,$$

so

$$\frac{x_1+\cdots+x_n}{n} = 25 > 21 \geq \sqrt{\frac{x_1^2+\cdots+x_n^2}{n}}\,.$$

By Theorem 1, this inequality is not true for any real values of x_1, x_2, x_3, x_4; in other words, the system of equations has no solution.

Example 3. *Let us find all real solutions to the following system of equations:*

$$-a_1 + a_2 - a_3 + \cdots + (-1)^n a_n = n \qquad \textit{and} \qquad a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2 = n\,.$$

Set $x_1 = -a_1$, $x_2 = a_2$, $x_3 = -a_3$, ..., $x_n = (-1)^n a_n$. Then

$$\frac{x_1+\cdots+x_n}{n} = \frac{-a_1+a_2-a_3+\cdots+(-1)^n a_n}{n} = \frac{n}{n} = 1$$

and

$$\sqrt{\frac{x_1^2+\cdots+x_n^2}{n}} = \sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{4}} = \sqrt{\frac{n}{n}} = \sqrt{1} = 1\,,$$

so

$$\frac{x_1+\cdots+x_n}{n} = \sqrt{\frac{x_1^2+\cdots+x_n^2}{n}}\,.$$

By Theorem 1, $x_1 = x_2 = \cdots = x_n = 1$; that is, $a_k = (-1)^k$ for $k = 1, \ldots, n$.