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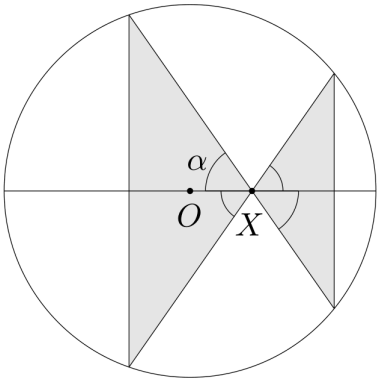
Problems 1571–1580

Q1571 Find positive integers a, b, c such that $a \leq b \leq c$ and

$$\frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} = \frac{5}{26}\,.$$

Q1572 In the following diagram, all four of the marked angles are equal.
Find the total area of the two shaded triangles in terms of the radius r of the circle, the distance $x_0 = OX$ and the angle α.



Q1573 The diagram below shows a rectangular grid with varying row heights and column widths. Six of the sub-rectangle areas are shown: find the value of x.

x		$x+1$
$x+2$	$x+3$	
	$x+4$	$x+5$

Q1574 Find the smallest positive integer n with the property that, if n is divided by 61, then the 21st digit after the decimal point is 1, and the 41st digit is 9.
(**Hint**. If 10^{20} is divided by 61, then the remainder is 13.)

Q1575 Let m and n be positive integers with $m \leq n$ and consider sequences

$$a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$$

of $2n$ coin tosses, where each a_k and each b_k is either Heads or Tails. How many of these sequences have exactly m more Heads among the b_ks than among the a_ks?

Q1576 This puzzle is inspired by www.mathsisfun.com/games/breaklock.html, a combination of Mastermind and the Android pattern lock. We have a square pattern of nine dots, here supplemented by letters for easy reference:

A code consists of four different dots connected by straight lines, for example $gdch$. Order is important: for example, $hcdg$ is different from $gdch$. It is not possible to skip over a dot which has not – or not yet – been used: for example, $agei$ and $agdc$ are illegal. However, it is permitted to skip over a dot that has already been used: for example, $aecg$ is allowed.

You guess the code $abcf$ and are told that two of those letters are part of the code and are in the correct position in your guess, while another one is correct but not in the correct position. You then guess $dghi$ and are told that one of these letters is correct and is in the correct position. You will be given similar information for future guesses. Can you find the code for certain in at most two more guesses?

Q1577 Is it possible to remove one of the numbers

$$1!, 2!, 3!, \ldots, 99!$$

in such a way that the product of the remaining 98 numbers is a square?
(**Note**: The exclamation mark denotes a factorial; for instance, $5! = 1{\times}2{\times}3{\times}4{\times}5 = 120$.)

Q1578 Find the points of inflection, and the tangents at these points, on the graph of

$$f(x) = x^4 - 2x^3 - 36x^2 + 28x + 99\,.$$

This is a routine problem if you have studied calculus, so do it **without** calculus.

Q1579 Consider the set

$$S = \{1, 3, 12, 16, 18, 32, 36, 108, 128, 144, 162, 192, 324, \ldots\}$$

of positive integers which have no prime factors except for 2 and 3, and which when written in base 10 have first digit 1 or 3. (For example, 15 begins with a 1 but is not in S since it is 3×5; similarly, 72 is also not in S, despite that $72 = 2 \times 2 \times 2 \times 3 \times 3$, since it begins with a 7.) We can form paths from numbers in S, by defining two numbers to be adjacent in the path if their quotient is 2 or 3 or 6. An example of a path is

$$12 \longrightarrow 36 \longrightarrow 18 \longrightarrow 108 \longrightarrow 324 \longrightarrow 162\,.$$

(a) Find a number in S which is not part of any path containing two or more elements.

(b) Show that any path with two or more elements can be continued indefinitely.

Q1580 Can you place ten consecutive integers in the circles below so that every line of four integers has the same sum?