Parabola Volume 54, Issue 3 (2018)

Solutions 1561–1570

Q1561 Let a, b, c be positive numbers for which

$$\frac{a+b}{c} = 2018 \quad \text{and} \quad \frac{b+c}{a} = 2019\,.$$

Evaluate $\dfrac{a+c}{b}$.

SOLUTION We have

$$\frac{a+b+c}{c} = \frac{a+b}{c} + 1 = 2019\,, \qquad \frac{a+b+c}{a} = \frac{b+c}{a} + 1 = 2020$$

and so

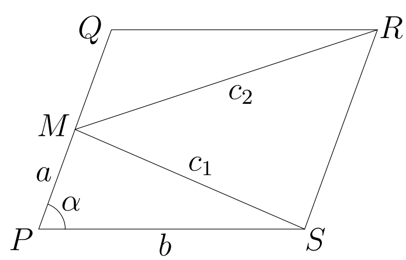
$$\frac{b}{a+b+c} = \frac{a+b+c}{a+b+c} - \frac{a}{a+b+c} - \frac{c}{a+b+c} = 1 - \frac{1}{2019} - \frac{1}{2020}\,.$$

Therefore,

$$\frac{a+c}{b} = \frac{a+b+c}{b} - 1 = \frac{1}{1 - \frac{1}{2019} - \frac{1}{2020}} - 1 = \frac{4039}{4074341}\,.$$

Q1562 In a parallelogram $PQRS$, let M be the midpoint of PQ.
Find the cosine of $\angle RMS$ in terms of the lengths PM and PS and the angle $\angle MPS$.

SOLUTION We write $a = PM$, $b = PS$ and $\alpha = \angle MPS$; also $c_1 = MS$ and $c_2 = MR$.



By the cosine rule, we have

$$\begin{aligned} c_1^2 &= a^2 + b^2 - 2ab\cos\alpha \\ c_2^2 &= a^2 + b^2 - 2ab\cos(\pi - \alpha) \\ &= a^2 + b^2 + 2ab\cos\alpha\,. \end{aligned}$$

Note that these equations give

$$c_1^2 + c_2^2 = 2a^2 + 2b^2 \quad \text{and} \quad c_1^2c_2^2 = (a^2+b^2)^2 - 4a^2b^2\cos^2\alpha\,.$$

Now since $PQRS$ is a parallelogram $RS = 2a$; using the cosine rule again,

$$4a^2 = c_1^2 + c_2^2 - 2c_1c_2\cos\beta\,,$$

where $\beta = \angle RMS$ is what we wish to find. Solving this equation and using the previous equations to eliminate c_1 and c_2 gives

$$\cos\beta = \frac{c_1^2 + c_2^2 - 4a^2}{2c_1c_2} = \frac{2b^2 - 2a^2}{2\sqrt{(a^2+b^2)^2 - 4a^2b^2\cos^2\alpha}}$$
$$= \frac{b^2 - a^2}{\sqrt{(a^2+b^2)^2 - 4a^2b^2\cos^2\alpha}}$$
$$= \frac{b^2 - a^2}{\sqrt{a^4 - 2a^2b^2\cos 2\alpha + b^4}}\,.$$

Q1563 Given a positive integer n, add the digits of n; then add the digits of the result; and so on, until you obtain a one–digit number. This one–digit number is called the *terminating sum* of n.[1] Find the terminating sum for

$$n = 2018^{2017^{2016^{\cdot^{\cdot^{\cdot^{3^{2^1}}}}}}}\,.$$

SOLUTION We write

$$m = 2017^{2016^{\cdot^{\cdot^{\cdot^{3^{2^1}}}}}} \quad \text{and} \quad p = 2016^{\cdot^{\cdot^{\cdot^{3^{2^1}}}}}\,,$$

and note for future use that

$$2018 = 9a + 2 \quad \text{and} \quad 2017 = 6b + 1$$

for certain integers a, b (in fact $a = 224$ and $b = 336$, though this is irrelevant). Now if two numbers differ by a multiple of 9 then their terminating sums are the same. By the Binomial Theorem, we have

$$n = (9a+2)^m = 9^m + \binom{m}{1}9^{m-1} \times 2 + \cdots + \binom{m}{m-1}9 \times 2^{m-1} + 2^m\,,$$

so n and 2^m differ by a multiple of 9 and we need to find the terminating sum of 2^m. The terminating sums of powers of 2 are

$$2\,,\ 4\,,\ 8\,,\ 7\,,\ 5\,,\ 1\,,\ 2\,,\ 4\,,\ 8\,,\ 7\,,\ 5\,,\ 1\,, \qquad (*)$$

with the same six values repeating indefinitely; so we need to find the remainder when m is divided by 6. Using the Binomial Theorem again,

$$m = 2017^p = (6b+1)^p = 6^p + \binom{p}{1}6^{p-1} + \cdots + \binom{p}{p-1}6 + 1\,,$$

which is a multiple of 6 with remainder 1. So the terminating sum of 2^m, and hence of n, is the first number in the list $(*)$, that is, 2.

Comment. The ideas behind this solution can be written much more simply when you have learned about *congruence arithmetic*.

[1]For more information on these sums, see the *Parabola* article
Terminating Sum of Digits of a Positive Integer by Sin Keong Tong.

Q1564 Write two numbers a, b in a row on a piece of paper. Form a list by writing their sum between them. Form another list by writing between every pair of adjacent numbers their sum. Repeat. For example, if $a = 1$ and $b = 2$, then we initially get

$$1\,, \quad 2\,.$$

Our first list is then

$$1\,, \quad 3\,, \quad 2\,,$$

our second list is

$$1\,, \quad 4\,, \quad 3\,, \quad 5\,, \quad 2\,;$$

and so on. What is the sum of the numbers in the nth list?

SOLUTION Let s_n be the sum of the nth list. Clearly $s_0 = a + b$. Given the numbers in one list, the numbers in the next list include all the same numbers, interspersed with extra numbers. These extra numbers are formed by adding all the previous numbers twice each, except for the first and last numbers which are used only once each. Therefore,

$$s_n = 3s_{n-1} - (a + b)\,.$$

To solve this, write $s_n = t_n + c$, where c is a constant. Then we have

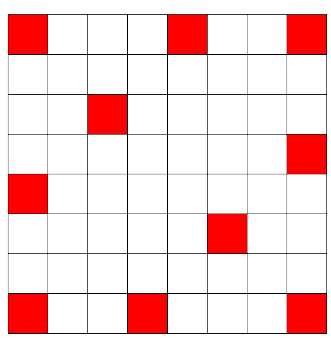
$$t_n = 3t_{n-1} + 2c - a - b\,.$$

Choosing $c = \frac{1}{2}(a+b)$ makes this very easy to solve: $t_n = 3t_{n-1}$, so $t_n = 3^n t_0$. Therefore, the sum of the numbers in the nth list is

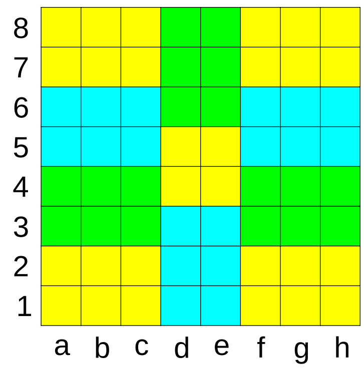
$$s_n = t_n + c = 3^n t_0 + c = 3^n(a + b - c) + c = \frac{(3^n + 1)(a + b)}{2}\,.$$

Q1565 Two squares on a (normal 8×8) chessboard are said to be *neighbours* if they can be reached from one another by means of at most two horizontal/vertical steps, **or** at most one horizontal/vertical and one diagonal step. Find the maximum number of squares that can be chosen on a chessboard such that no two are neighbours.

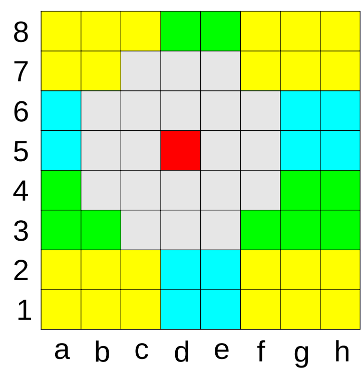
SOLUTION It's easy to get 10 by trial and error, as, for example, in the diagram.



To show that 11 is impossible, divide the board into eleven blocks as shown by the coloured regions.



It is easy to see that two squares in the same region must be neighbours; so, to get 11 squares for which no two are neighbours, we must take exactly one in each block. By symmetry, we may assume that we choose the square at d5. This rules out the squares shown in grey in the next diagram.



Finally, we can't take a5 as it eliminates all the remaining squares in the green block on the left hand side; so we must take a6; the only available square in the top left yellow block is now c8; and this makes it impossible to choose either of the green squares along the top edge. So, it's impossible to choose 11 squares such that no two are neighbours.

Q1566 Let m and n be positive integers with $m \neq n$. Prove that $m^4 + 3n^4$ can be written as the sum of the squares of three non–zero integers.

SOLUTION The coefficient of n^4 suggests that we try something like this:

$$m^4 + 3n^4 = (n^2 + a)^2 + (n^2 + b)^2 + (n^2 + c)^2 \,,$$

where a, b, c are expressions in terms of m, n. Expanding, we get

$$m^4 + 3n^4 = 3n^4 + 2(a + b + c)n^2 + a^2 + b^2 + c^2 \,.$$

Now somehow we have to get an m^4 term on the right hand side. It clearly can't come from the terms containing n, so it looks like one of the terms a^2, b^2, c^2 should be m^4. It obviously doesn't matter which one we go for, so let's say $c^2 = m^4$. Taking $c = m^2$ doesn't seem to lead anywhere (try it!), so we explore $c = -m^2$. This gives

$$m^4 + 3n^4 = m^4 + 3n^4 + 2(a + b - m^2)n^2 + a^2 + b^2 \,.$$

If we remove the a and b from the middle term by choosing $b = -a$ we have

$$m^4 + 3n^4 = m^4 + 3n^4 - 2m^2n^2 + 2a^2 \,,$$

and it is now clear that we get what we want by taking $a = mn$. So, to sum up, we have found that

$$m^4 + 3n^4 = (n^2 + mn)^2 + (n^2 - mn)^2 + (n^2 - m^2)^2 \,;$$

it is easy to check this by multiplying out the right hand side and, since m, n are unequal positive integers, it is clear that each bracketed term on the right hand side is a non–zero integer.

Q1567 Given a positive integer $n \geq 2$, find unequal real numbers a, b, **not** integers, such that

$$a - b \,,\; a^2 - b^2 \,, \ldots ,\; a^n - b^n$$

are all integers.

SOLUTION We have

$$b = \frac{(a+b) - (a-b)}{2} = \frac{1}{2}\Big(\frac{a^2 - b^2}{a - b} - (a - b)\Big) \,.$$

Since $a^2 - b^2$ and $a - b$ are integers, this is a rational number; and $x = a - b$ is an integer; so we can write

$$b = \frac{y}{z} \quad \text{and} \quad a = x + \frac{y}{z} \,,$$

where x, y, z are integers with $z > 0$. Expanding by the Binomial Theorem,

$$a^m - b^m = x^m + \binom{m}{1} x^{m-1}\frac{y}{z} + \cdots + \binom{m}{k} x^{m-k}\Big(\frac{y}{z}\Big)^k + \cdots + \binom{m}{m-1} x \Big(\frac{y}{z}\Big)^{m-1} \,.$$

Considering all these expressions for $m = 1, 2, \ldots, n$, the maximum power of z in any denominator is z^{n-1}; and every term has at least one x in the numerator. So, if x is a multiple of z^{n-1}, then every expression $a^m - b^m$ will be a sum of integers, and therefore an integer. Specifically, choose $y = 1$ (may as well keep things simple!) and $z = 2$ (to make sure that a and b are not integers) and $x = 2^{n-1}$. Then we have

$$a^m - b^m = 2^{m(n-1)} + \binom{m}{1} 2^{(m-1)(n-1)-1} + \cdots + \binom{m}{k} 2^{(m-k)(n-1)-k} + \cdots + \binom{m}{m-1} 2^{n-m} \,.$$

In this expression every term has $k \leq m - 1 \leq n - 1$; so

$$(m - k)(n - 1) - k \geq (n - 1) - k \geq 0 \,.$$

In other words, the powers of 2 in the expression always occur with non–negative exponents; therefore they are integers; therefore $a^m - b^m$ is an integer; and this is what we wanted. Our solution (one of many possibilities) is

$$a = 2^{n-1} + \frac{1}{2}, \quad b = \frac{1}{2}.$$

Q1568 Draw the graph of $\sin(y + |y|) = \sin(x + |x|)$.

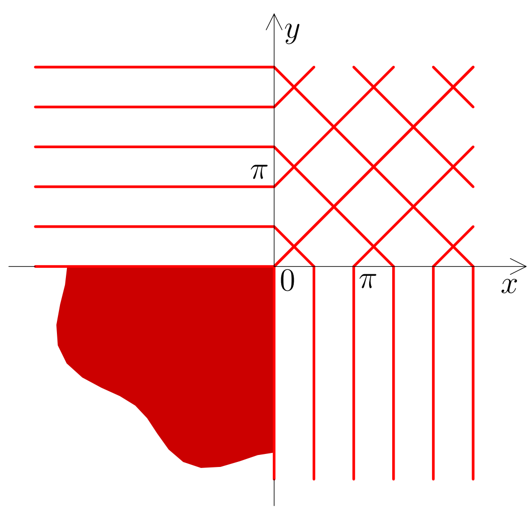
SOLUTION Remember that $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x \leq 0$. In the first quadrant (including the axes) we have $x \geq 0$, $y \geq 0$; so the equation becomes $\sin 2y = \sin 2x$. This is equivalent to

$$2y = 2x + 2k\pi \quad \text{or} \quad 2y = -2x + (2k+1)\pi, \qquad k \in \mathbb{Z},$$

that is, a set of parallel lines $y = x + k\pi$ of gradient 1, together with a set of parallel lines $y = (k + \frac{1}{2})\pi - x$ of gradient -1. The other quadrants are easier:

- in the second quadrant we have $x \leq 0$, $y \geq 0$; so we have $\sin 2y = 0$, which is a set of horizontal lines $y = \frac{1}{2}k\pi$;
- in the third quadrant we have $x \leq 0$, $y \leq 0$; so the equation is $0 = 0$, which is true for **all** points in the third quadrant;
- in the fourth quadrant we have $x \geq 0$, $y \leq 0$; so $0 = \sin 2x$, a set of vertical lines $x = \frac{1}{2}k\pi$.

So the graph is as shown (including the negative halves of the x and y axes, but not the positive halves).



Q1569 We have a row of n coins. A "move" consists of selecting a coin which is tails up, and turning over both that coin and the one (if any) immediately to its left. An example of a sequence of three moves involving five coins is

$$HTTTT \to HTTHH \to THTHH \to HHTHH.$$

Prove that if we are allowed to choose the initial arrangement of coins, then it is possible to make $\frac{1}{2}n(n+1)$ moves before getting stuck; but that it is never possible to make more than this many moves.

SOLUTION Give a coin in position k from the left a value of k if it is tails (and nothing if it is heads). Give each arrangement of coins a value which is the sum of the values of its individual coins. For example, the four positions above have values $14, 5, 4, 3$. A move changes the value of a position as follows:

- if the selected coin is in position 1 (the left–hand end of the row), turning this coin from tails to heads decreases the value of the position by 1;
- if the selected coin is tails in position $k > 1$ and the coin in position $k-1$ is heads, then the move loses value k in position k but gains $k-1$ in position $k-1$, for an overall decrease in value of 1;
- if the selected coin is tails in position $k > 1$ and the coin in position $k-1$ is tails, then the move decreases the value of the position by $2k-1$.

Now the "most valuable" position is a row of tails, with score

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)\,,$$

and the least valuable is a row of heads with score 0, at which point we are stuck and cannot make any further moves. Since, at best, the value of a position decreases by 1 at each step, it is impossible to make more than $\frac{1}{2}n(n+1)$ moves. To achieve this many moves, we must start with a row of tails and avoid moves that decrease the position value by more than 1. This can be done by always selecting the leftmost tail: it will then have a head (or nothing at all) on its left, and the move will decrease the value by 1.

Q1570 Find all solutions of the simultaneous equations

$$2x = z(3x^2 + 3y)\,, \quad 2 = z(3x + 3y^2)\,, \quad x^3 + 3xy + y^3 = 5\,.$$

SOLUTION The first equation minus x times the second gives

$$0 = 3zy(1 - xy)\,.$$

The second equation in the question implies that $z \neq 0$, so either $y = 0$ or $xy = 1$.

- If $y = 0$ then the third equation gives $x = \sqrt[3]{5}$ and either of the others gives $z = 2/3x$.
- If $xy = 1$ then the third equation gives $x^3 + y^3 = 2$. Multiplying by x^3, rearranging and using $xy = 1$ again gives $x^6 - 2x^3 + 1 = 0$; this can be factorised as $(x^3 - 1)^2 = 0$ and so the only (real) solution is $x = 1$; then we have $y = 1$ and $z = 2/(3x + 3y^2)$.

Therefore the solutions are

$$x = \sqrt[3]{5}\,,\ y = 0\,,\ z = \frac{2}{3\sqrt[3]{5}} \quad \text{and} \quad x = 1\,,\ y = 1\,,\ z = \frac{1}{3}\,.$$