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PYTHAGOREAN PATTERNS

Most of you know what is meant by a Pythagorean triple. It is a set of three natural numbers $\{a,b,c\}$ such that $a^2 + b^2 = c^2$. For example the following are Pythagorean triples:

$$\{3,4,5\}, \quad \{5,12,13\}, \quad \{7,24,25\}, \quad \{6,8,10\}, \quad \{10,24,26\}.$$

Let us see if we can extend our list of triples by using number patterns. There is something similar about the first three triples in the above listing. Let us list them vertically in an array:

3	5	7
4	12	24
5	13	25 .

We notice that each entry in the third row is one greater than the corresponding entry in the second row. Also we may observe that

$$3^2 = 4 + 5,$$
$$5^2 = 12 + 13,$$
$$7^2 = 24 + 25.$$

Let us try this for 9 and 11.

$$9^2 = 81 = 40 + 41$$
$$11^2 = 121 = 60 + 61.$$

When we do the necessary arithmetic we find that 9,40,41 and 11,60,61 are indeed Pythagorean triples. So the array now becomes

3	5	7	9	11
4	12	24	40	60
5	13	25	41	61

and it looks as though we may have a method of finding triples in which the first entry is odd. Also once we have extended the array this far we can notice patterns in the second and third rows. What will be the next entry in the second row? Look at the differences between the entries.

4 ↷ 12 ↷ 24 ↷ 40 ↷ 60 . . .

8 ↷ 12 ↷ 16 ↷ 20

4 4 4

In each case the difference is 4 greater than the previous difference. So we could guess that the next difference will be 24. Thus the next entry in the second row will be 84 and the one after that will be 112.

The array now becomes

3	5	7	9	11	13	15
4	12	24	40	60	84	112
5	13	25	41	61	85	113

and our task of finding triples appears easier. However, it is well to pause a while and consider what we are doing. In each case we have seen a pattern, then tested to see whether it works for the next couple of numbers. Fortunately each time we've tested the pattern it has worked. Surely, though, there must remain a nagging doubt as to whether it will work for *all* such numbers. What we need is an algebraic verification of the result. (Algebra enables us to write a pattern in one line.) We need to show it will be a true result for any number we choose. To show its truth in all cases we show its truth for an arbitrary number "n".

What is the first pattern we used? It says to take an odd number $(2n + 1)$. Square it $(4n^2 + 4n + 1)$. Form the other two numbers of the triple by subtracting ½ from half of the square, and adding ½ to half of the square. The triple, in terms of "n", is therefore $2n + 1$, $2n^2 + 2n$, $2n^2 + 2n + 1$. For this to be a Pythagorean triple we have to show that

$$(2n^2 + 2n + 1)^2 = (2n^2 + 2n)^2 + (2n + 1)^2.$$

You may care to check that this is in fact a true result. The doubt that existed before about its truth for any odd number should now be removed.

The second pattern used enabled us to write the entries in the second (and hence the third) rows. You may like to attempt an algebraic verification of the fact that the difference between an entry in the second row and the one before it (in the second row) is always $4n$.

Before leaving the odd numbers let us return to the array to observe another pattern.

3 ↓	← 5	7	9 ↓	← 11	13	15 ↓	← 17
4	→ 12	24	40	→ 60	84	112	→
5	13	25	41	61	85	113	

Noting the direction of the arrows we can see that $5 + 3 + 4 = 12$, $7 + 5 + 12 = 24$ and so on. It looks as though we have another way of finding the entries in the second row. $17 + 15 + 112 = 144$ so the next triple appears to be 17,144,145 . A little arithmetic soon verifies that this is indeed a triple. Of course we ought to verify this algebraically also. Are there any other patterns in the array?

What about beginning an array with even numbers? We can't find a Pythagorean triple beginning with 2 so our array will have to begin at 4.

4	6	8	10	.	.	.
3	8	.	.			
5	10	.	.			

Where do we begin? We note that this time the difference between the corresponding entries in the second and third rows is 2. Earlier we had some success with $3^2 = 4 + 5$ so let's look at 4^2 and 6^2. We note that $4^2 = 2 \times (3 + 5)$ and $6^2 = 2 \times (8 + 10)$. Without too much trouble we can determine that $8^2 = 2 \times (15 + 17)$ and $10^2 = 2 \times (24 + 26)$. Let's write them in (after checking that they are Pythagorean triples).

4	6	8	10
3	8	15	24
5	10	17	26

When we do the algebra involved we find that the triples in this case are all of the form, $2n$, n^2-1 and $n^2 + 1$. ($n \neq 1$). Now we can set off on another round of pattern searching and verification. Good hunting!

By the way how many Pythagorean triples can you find in which the smallest number is 15? 45?

What about seeing what you can find out about the Pythagoreans. Write in and let us know.

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