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HOW TO TRISECT AN ANGLE

Among the geometrical problems which have intrigued amateurs throughout the ages, the geometrical trisection (i.e. division in three equal parts) of an angle ranks high in popularity. Although it has been known since the middle of the last century that it is impossible to trisect most angles by the use of a straight edge and compasses alone (the right angle being a notable exception; can you trisect it?) there are still many who attempt the impossible, and hardly a year passes by without a solution being offered by well meaning and sometimes geometrically quite well educated and ingenious amateurs.

It is easy to understand why so many still hope for a miraculous breakthrough. If an old problem which has been around for some time is suddenly settled and a proof is found, most people accept it without grudge, even if they themselves cannot follow the intricacies of the argument. But if something is proved to be impossible then the proof is often received with scepticism. Didn't mathematicians "prove" at the turn of the century that machine propelled flight (other than gas-filled balloons) is impossible? Yet aeroplanes fly merrily all over the world. Of course in examples such as this it is not the mathematical reasoning but the mathematical model of the physical object which is usually at fault. No such spurious model-making is involved in the impossibility proof of the trisection problem.

The proof rests on the observation that trisection of a given angle a is equivalent to the construction of a distance of length $\sin\frac{a}{3}$ when a distance of length $\sin a$ is given. By simple trigonometry it can be shown that the quantity $\sin\frac{a}{3}$ satisfies the cubic equation

$$4x^3 - 3x + \sin a = 0$$

(set $a = 3\beta$ and prove that $\sin 3\beta = 3\sin\beta - 4\sin^3\beta$). Now it turns out that roots of such a cubic equation (for given $\sin a$) cannot be constructed by ruler and compasses unless a is given some very special values. Demonstration of this fact goes back to the work of one of the most ingenious mathematicians of the last century, Evariste Galois. [You may want to read his fascinating and tragic life story in E.T. Bell's "Men of Mathematics".] Unfortunately it is too involved to be reproduced here; the argument is remotely similar to (though far more complicated than) the well known proof that the roots of the equation $x^2 - 2$, namely $\pm\sqrt{2}$, cannot be expressed as a rational fraction. That certain angles cannot be trisected, was already known to Gauss; he showed for instance that the regular 9-sided polygon cannot be constructed, and hence the angle 120° cannot be trisected. Of course the proof presupposes that only the usual ruler and compasses operations are permitted. For instance if the ruler has two markings, say at unit distance apart, then any angle can be trisected with it (and compasses). This was already known to Pappus and Archimedes some 2000 years ago. The following is a beautiful method of trisection due to Archimedes.

Let AOB be the angle to be trisected. With O as centre and radius unity, draw a semicircle ABC. Slide the marked ruler through B until it intercepts the circle and the diameter AC produced in points D and P such that these points coincide with the markings on the ruler. Then D and P are unit distance apart and the angle BPA is one-third of the angle AOB. (Prove it.)

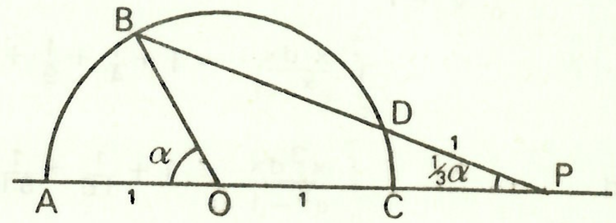

Figure 1

If only conventional straight edge and compasses constructions are allowed then trisection cannot of course be exact for all angles, but very good approximations can nevertheless be achieved. An example of a remarkably accurate (inexact) construction is given in H. Tietze's book *Famous problems of mathematics* (Graylock Press, 1965). Chapter III deals with the trisection problem and the construction is given on p. 55.

Problems

1. Prove that $\sin 3\beta = 3 \sin \beta - 4 \sin^3 \beta$.
2. Show that the construction of Archimedes is correct, that is,

$$\angle BPA = \tfrac{1}{3} \angle AOB \quad \text{in Fig. 1.}$$

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Little Boy Bewildered

"I don't know," said a little boy, "if I add nothing to a number, it stays the same. If I subtract nothing from a number, it stays the same. If I multiply the number by zero, then I don't multiply it by anything at all. Surely if I don't multiply a number by anything at all, the number remains the same."