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# THE SERIES $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots + \frac{1}{r^2} + \ldots$

In an appendix to a Physics book, I found the derivation of infinite integrals of the type $\int_0^\infty \frac{x^n\,dx}{e^x - 1}$. The book showed that

$$\int_0^\infty \frac{x\,dx}{e^x-1} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots + \frac{1}{r^2} \ldots$$

and

$$\int_0^\infty \frac{x^3\,dx}{e^x-1} = 1 + \frac{1}{16} + \frac{1}{81} + \ldots + \frac{1}{r^4} + \ldots$$

Here the derivation finished and it mentioned the mysterious result that these sums were $\pi^2/6$ and $\pi^4/90$. After much labour, I have found the following method for evaluating these series.

Since $-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$

$$-\int \frac{\log(1-x)}{x}\,dx = \int \left(1 + \tfrac{1}{2}x + \tfrac{1}{3}x^2 + \ldots\right) dx$$

$$= x + \tfrac{1}{4}x^2 + \tfrac{1}{9}x^3 + \ldots + \tfrac{1}{r^2}x^r + \ldots \qquad (1)$$

We now substitute $x = e^{i\theta} = \cos\theta + i\sin\theta$. Thus $\frac{dx}{d\theta} = i\,e^{i\theta} = ix$ and

$$\log(1-x) = \log(1-e^{i\theta}) = \log(1-\cos\theta - i\sin\theta) = \log re^{i\phi}$$

$$= \log r + i\phi$$

where $r^2 = (1-\cos\theta)^2 + \sin^2\theta = 2-2\cos\theta$

and $\tan\phi = \frac{\sin\theta}{\cos\theta - 1} = \tan(\frac{1}{2}\theta - \frac{1}{2}\pi)$.

Therefore $\phi = \frac{1}{2}\theta - \frac{1}{2}\pi$ and

L.H.S. of (1) $= i\int \log(1-e^{i\theta})d\theta = i\int(\log r + i\phi)\,d\theta = -\frac{1}{2}\int(\theta-\pi)d\theta + i\int \log r\,d\theta$

Also by de Moivre's theorem

$$x^r = (\cos\theta + i\sin\theta)^r = \cos r\theta + i\sin r\theta.$$

Substituting these results in equation (1) and equating real parts, we get

$$\sum_{r=1}^{\infty} \frac{\cos r\theta}{r^2} = \tfrac{1}{2}\int(\theta-\pi)\,d\theta = \tfrac{1}{4}\theta^2 - \tfrac{1}{2}\pi\theta + c_2 \qquad (2)$$

where $c_2$ is a constant.

When $\theta = 0$, $c_2 = 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{r^2} + \ldots$

$$\begin{aligned} \text{When } \theta = \pi, \quad -\tfrac{1}{4}\pi^2 + c_2 &= -1 + \tfrac{1}{4} - \tfrac{1}{9} + \tfrac{1}{16} - \ldots \\ &= -(1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \ldots) + 2(\tfrac{1}{4} + \tfrac{1}{16} + \tfrac{1}{36} + \ldots) \\ &= -c_2 + \tfrac{1}{2}(1 + \tfrac{1}{4} + \tfrac{1}{9} + \ldots) \\ &= -\tfrac{1}{2}c_2. \end{aligned}$$

So $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{r^2} \ldots = c_2 = \pi^2/6.$

To get the general result, integrate equation (2) to get

$$\sum_{r=1}^{\infty} \frac{\sin r\theta}{r^3} = \int (\tfrac{1}{4}\theta^2 - \tfrac{1}{2}\pi\theta + \frac{\pi^2}{6})\, d\theta = \frac{\theta^3}{2 \times 3!} - \frac{\pi\theta^2}{2 \times 2!} + \frac{\pi^2\theta}{6} + c_3.$$

When $\theta = 0$, $c_3 = 0$.

The following general pattern is emerging:

$$c_k = 0 \text{ for k odd and } c_{2n} = (-1)^{n-1} \sum_{r=1}^{\infty} 1/r^{2n}$$

where $c_{2n}$ can also be found by putting $\theta = \pi$ in equation (2) after integrating 2(n–1) times.

The odd case $\sum\limits_{r=1}^{\infty} 1/r^{2n+1}$ depends on equating imaginary parts and finding

$\log r\, d\theta$ which I have not done.

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[Editor's Note: David's article was originally a lot longer, but because of the difficulty of typing mathematics I have left some of his details out which you can fill in for yourselves or find in a textbook.]

The "mysterious" formula $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} + \ldots = \frac{\pi^2}{6}$ was discovered by the famous Swiss mathematician Leonhard Euler (1707 - 1783), in whose hands Newton's calculus blossomed into modern mathematical analysis. This particular formula was one of the many precious gems found by the master.

It is worth recording how Euler himself came to the formula. Take the function $(\sin \pi x)/\pi x$ where of course the angle $\pi x$ is not in degrees but in radians. This function has the following two properties: First, its "value" at $x = 0$ is 1. More precisely, since at $x = 0$ both the numerator and the denominator vanish therefore the expression is meaningless, the function $(\sin \pi x)/\pi x$ tends to 1, or has the limit 1, when x approaches 0. The second property is that $(\sin \pi n)/\pi n = 0$ for all non-zero integers n, that is the function is zero for $x = 1$, $x = -1$, $x = 2$, $x = -2$ etc.

Using these two properties Euler argued that $(\sin \pi x)/\pi x$ may be factorized, just as if it were a polynomial, into the product of infinitely many factors as follows:

$$(\sin \pi x)/\pi x = (1-x)(1+x)(1-\tfrac{x}{2})(1+\tfrac{x}{2})(1-\tfrac{x}{3})(1+\tfrac{x}{3})\ldots$$

$$= (1-x^2)(1-\tfrac{x^2}{4})(1-\tfrac{x^2}{9})\ldots(1-\tfrac{x^2}{n^2})\ldots \qquad (*)$$

Now multiply out the product, as if it had only a finite number of factors; to get

$$1-x^2-\tfrac{x^2}{4}+\tfrac{x^4}{4}-\tfrac{x^2}{9}+\tfrac{x^4}{9}+\tfrac{x^4}{36}-\tfrac{x^6}{36}+\ldots \text{ etc.}$$

rearrange the terms and collect all those which contain $x^2$, then those which contain $x^4$, etc.

The coefficient of $x^2$ on the right hand side is just the expression

$-(1+\frac{1}{4}+\frac{1}{9}+\ldots+\frac{1}{r^2}+\ldots)$ and so

$$\sin \pi x - \pi \Sigma \tfrac{1}{r^2} x^3 + \ldots$$

Differentiating three times and putting $x = 0$, we get Euler's formula. Similarly, by differentiating twice again and using a bit of clever manipulation one can derive $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ and more general expressions for $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ when $k = 3, 4$ etc.

You will admit that a lot of handwaving was needed in the above calculations and it is not easy to make the argument quite precise (although it can be done). Euler himself found later several proofs which were completely correct, but neither he nor others were ever able to discover a simple expression for $\sum_{n=1}^{\infty} \frac{1}{n^3}$ and generally for sums of odd powers of $\frac{1}{n}$. It is hardly surprising that David was also defeated in his search for such an expression.