270. Find all positive integers between 1 and 100 having the property that $(n-1)!$ is not divisible by n^2.

271. Prove that if the sum of the fractions $\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}$ (where n is a positive integer) is put in decimal form, it forms a non-terminating decimal which is periodic after several terms.
(e.g. For $n = 3, \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60} = .78\dot{3}$ is periodic after 2 decimal places.)

272. Let m and n be two relatively prime positive integers. Prove that if the $m + n-2$ fractions

$$\frac{m+n}{m}, \frac{2(m+n)}{m}, \frac{3(m+n)}{m}, \ldots, \frac{(m-1)(m+n)}{m},$$

$$\frac{m+n}{n}, \frac{2(m+n)}{n}, \frac{3(m+n)}{n}, \ldots, \frac{(n-1)(m+n)}{n},$$

are plotted as points on the real number line, exactly one of these fractions lies inside each of the unit intervals (1,2), (2,3), (3,4), . . . , $(m+n-2, m+n-1)$. (e.g. If $m = 3$, $n = 4$, then 7/4 is between 1 and 2, 7/3 is between 2 and 3, 14/4 is between 3 and 4, 14/3 is between 4 and 5, and 21/4 is between 5 and 6.)

Solutions to Problems J251–O260 (Vol. 10 No. 3)

Junior

J251 Farmer Jones grew a square number of cabbages last year. This year he grew 41 more cabbages than last year and still grew a square number of cabbages. How many did he grow this year?

Answer: If Farmer Jones grew x^2 cabbages this year, and y^2 last year then $x^2-y^2 = 41$.

i.e. $(x-y)(x + y) = 41$. Since the only factorisation of 41 into two integer factors is 1×41 we must have $x-y = 1$, $x + y = 41$, and solving gives $x = 21$, $y = 20$. Hence $x^2 = 441$.

J252 I met triplets, A, B and C whose names were John, Peter, and Mick. When I asked who was who, A answered, "I'm not Peter."

B said, "I'm Peter."

C said, "I'm not John."

Then they told me that only one of them was telling the truth. Who was who?

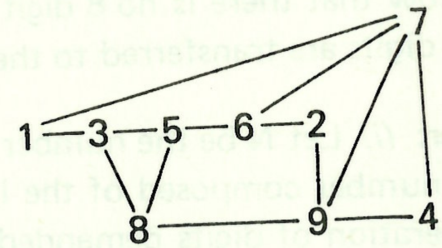
B must be Mick. We now have a contradiction since A has to be Peter, making his statement false after all. This contradiction shows that A's statement cannot have been true.

Therefore A is Peter. B's statement is then clearly also a lie, making C's statement the only true one. Hence C must be Mick and B must be John.

Intermediate

I253 Is it possible to write the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9 on a circle in such an order that the sum of two neighbouring numbers is never divisible by 3, 5, or 7?

Answer: In the diagram a line has been drawn joining 2 digits if their sum is not divisible by 3, 5 or 7. It is easy to find the following circuit of lines which visits each digit once:– 1,3,8,5,6,2,9,4,7,(1).



I254 Solve the following simultaneous equations for x and y

$$xy(x-y) = ab(a-b)$$
$$x^3-y^3 = a^3-b^3.$$

Answer: $$xy(x-y) = ab(a-b) \qquad (1)$$
$$x^3-y^3 = a^3-b^3 \qquad (2)$$

Subtracting 3 times the first equation from the second gives

$$x^3-3x^2y+3xy^2-y^3 = a^3-3a^2b+3ab^2-b^3$$

or $(x-y)^3 = (a-b)^3$, whence

$$x-y = a-b. \qquad (3)$$

If $a \neq b$, dividing (3) into (1) gives

$$xy = ab. \qquad (4)$$

Squaring (3) and adding 4 x (4) yields

$$(x+y)^2 = (x-y)^2 + 4xy = (a-b)^2 + 4ab = (a+b)^2$$

so $$x+y = a+b \qquad (5)$$

or $$x+y = -(a+b). \qquad (6)$$

Simultaneous solution of (3) and (5) yields $x = a;\ y = b;$ and of
(3) and (6) yields $x = -b;\ y = -a.$

These are the only 2 real solutions unless it happens that $a = b$. In this case any pair (x,y) with $x = y$ is a solution.

[The "complex number" pairs $(x,y) = (\omega a, \omega b);\ (\omega^2 a, \omega^2 b);\ (-\omega b, -\omega a)$; and $(-\omega^2 b, -\omega^2 a)$, where $\omega = -\frac{1}{2} + i.(\frac{1}{2}\sqrt{3})$, also satisfy the equations. In fact, if complex numbers are considered (3) should be replaced by $x-y = a-b$ OR $x-y = \omega(a-b)$ OR $x-y = \omega^2(a-b)$ with appropriate modification of the rest of the working.]

I255 (i) Find a 6-digit number which is multiplied by the factor 6 if the final 3 digits are removed and placed (without changing their order) at the beginning.

(ii) Show that there is no 8 digit number which is increased by the factor 6 if the final 4 digits are transferred to the beginning.

Answer: *(i)* Let N be the number, X the number composed of its first 3 digits and Y the number composed of the last 3 digits. Then $N = 1000X + Y$. After making the alteration of digits demanded, the number $1000Y + X$ is obtained and this is 6N. Thus

$$\begin{aligned} 1000Y + X &= 6(1000X + Y) \quad \text{which yields} \\ 994Y &= 5999X \quad \text{and (after cancelling a factor of 7)} \\ 142Y &= 857X. \end{aligned}$$

A solution in 3-digit numbers is obviously $Y = 857$, $X = 142$. In fact, since 142 and 857 are relatively prime, it is the only such solution; other integer solutions are given by $Y = 857k$, $X = 142k$ for any integer k, but if $k > 1$, Y is no longer a 3 digit number.

Hence 142, 857 has the required property.

(Note the similarity with the decimal expansion for 1/7. Can anyone see why?)

(ii) Similar working gives $9994Y = 59{,}999X$ where X and Y are 4-digit numbers. However, this time 9994 and 59,999 are already relatively prime (i.e. have no common factor) so Y must be divisible by 59,999 which is not possible if Y has only 4 digits.

Open

O256 Simplify the following expression

$$(1+\tfrac{1}{3})(1+\tfrac{1}{9})(1+\tfrac{1}{81})(1+\tfrac{1}{3^8})\ \ldots\ (1+\tfrac{1}{3^{2^n}}).$$

Answer: If $X = (1+\frac{1}{3})(1+\frac{1}{9})(1+\frac{1}{81}) \ldots (1+\frac{1}{3^{2^n}})$

then $\frac{2}{3}x = (1-\frac{1}{3})X = (1-\frac{1}{3})(1+\frac{1}{3})(1+\frac{1}{9})(1+\frac{1}{81}) \ldots (1+\frac{1}{3^{2^n}})$

$= (1-\frac{1}{9})(1+\frac{1}{9})(1+\frac{1}{81}) \ldots (1+\frac{1}{3^{2^n}})$

....

$= (1-\frac{1}{3^{2^n}})(1+\frac{1}{3^{2^n}}) = (1-\frac{1}{3^{2^{n+1}}})$

Thus $X = \frac{3}{2}(1-\frac{1}{3^{2^{n+1}}})$.

O257 The equation $x^5 + y^2 = z^4$ has $x = 2$, $y = 7$, $z = 3$ as a solution in integers. Are there any other solutions in integers?

Answer: Obviously, if x_0, y_0, z_0 is one solution and m is any integer then

$$x = x_0.m^4;\ y = y_0.m^{10};\ z = z_0.m^5$$

is also a solution. This gives an infinite set of solutions related to $(x_0,y_0,z_0) = (2,7,3)$.

However, other solutions also exist. Since $x^5 = z^4 - y^2 = (z^2+y)(z^2-y)$, z^2+y and z^2-y must be factors of x^5. One possibility is

$$z^2 - y = x^2$$

$$z^2 + y = x^3.$$

Solving these gives $y = ½(x^3-x^2)$, $z^2 = ½(x+1)x^2$. Now $½(x+1)x^2$ is a perfect square if and only if $½(x+1)$ is. Thus if $½(x_0+1) = k^2$ (or $x_0 = 2k^2-1$), $z_0 = kx_0 = k(2k^2-1)$ and $y_0 = ½(x_0{}^3 - x_0{}^2) = (2k^2-1)(k^2-1)$ for any integer k, then (x_0,y_0,z_0) is a solution of $x^5 + y^2 = z^4$. This infinite family does not include (2,7,3).

Another infinite set of solutions may be found by putting $y = bx^2$, $z = ax$, where a,b are position integers. Then

$$a^4x^4 = z^4 = x^5 + y^2 = x^5 + b^5x^4.$$

So $x = a^4 - b^2$, $y = bx^2 = b(a^4-b^2)^2$ and $z = ax = a(a^4-b^2)$.

Can any readers find any more solutions?

0258 Show that if p is an odd prime, it divides the difference

$$[(\sqrt{5}+2)^p] - 2^{p+1}.$$

Note that $[x]$ means the integer n such that $n \leqslant x < n+1$.

Answer: By the binomial expansion

$$x = (\sqrt{5}+2)^p - (\sqrt{5}-2)^p = 2\,({}^pC_1.5^{\frac{1}{2}(p-1)}.2 + {}^pC_3.5^{\frac{1}{2}(p-3)}.2^3 + \ldots + {}^pC_{p-2}.5.2^{p-2}) + 2^{p+1}$$

is an integer. Since $0 < (\sqrt{5}-2)^p < 1$, x is the integer $[(\sqrt{5}+2)^p]$. Hence

$$[(\sqrt{5}+2)^p] - 2^{p+1} = x - 2^{p+1} = 2\,({}^pC_1.5^{\frac{1}{2}(p-1)}.2 + {}^pC_3.5^{\frac{1}{2}(p-3)}.2^3 + \ldots + {}^pC_{p-2}.5.2^{p-2}) \qquad (1)$$

where ${}^pC_k = \frac{p.(p-1)\ldots(p-k+1)}{1.2.3\ldots k}$ is an integer. In cancelling the denominator into the numerator, no factor divides the prime number p. Hence p divides ${}^pC_k$ when $k < p$. Thus in the formula (1), every term on the R.H.S. is divisible by p, which establishes the result.

0259 *(Submitted by M. Durie)* You have a beam balance, but no weights, and a collection of 12 similar coins, one of which however is counterfeit. It is a different weight from the good coins, but you do not know whether it is heavier or lighter. Locate the bad coin and its relative weight in three weighings.

Answer: Label the coins 1,2,3, . . . , 12.

First weighing (1,2,3,4) v (5,6,7,8).

Case 1: It balances. The counterfeit coin is one of 9, 10, 11, 12.

Second weighing (1,2,3) v (9,10,11).

If it balances, the bad coin is 12, and a third weighing (1) v (12) will decide whether it is light or heavy.

If it does not balance, the bad coin is one of 9, 10, or 11 and we know whether it is too heavy or too light. After a third weighing (9) v (10) the counterfeit coin may be identified.

Case 2: We may assume (altering the labels on the coins if necessary) that (1,2,3,4) is heavier than (5,6,7,8).

Second weighing (4,5,6,7) v (8,9,10,11).

If (4,5,6,7) is heavier than (8,9,10,11) the bad coin is either 4 (heavy) or 8 (light). Compare either of these with 1.

If (4,5,6,7) balances (8,9,10,11), the bad coin is one of 1,2,3 and is heavy. Weigh (1) v (2) for the third weighing.
If (4,5,6,7) is lighter than (8,9,10,11), the bad coin is one of 5,6,7 and is light. Weigh (5) v (6) for the third weighing.

O260 If n and k are any positive integers, show that

$$\frac{1}{n} - {}^kC_1 \frac{1}{n+1} + {}^kC_2 \frac{1}{n+2} + \ldots + (-1)^j\, {}^kC_j \frac{1}{n+j} + \ldots + (-1)^k \frac{1}{n+k}$$

is equal to $\frac{1}{l.c.m.\{D_0, D_1, \ldots, D_k\}}$ where D_j is the denominator when $\frac{{}^kC_j}{n+j}$ is put in lowest terms, $(j = 0, 1, 2, \ldots, k)$. $l.c.m.\{D_0, D_1, \ldots, D_k\}$ means the smallest number which $D_0, D_1, \ldots, D_k$ will all divide.

Answer: If we denote the given expression by $f_k(n)$, we see that

$f_0(n) = \frac{1}{n}$

$f_1(n) = \frac{1}{n} - \frac{1}{n+1} = f_0(n) - f_0(n+1)$

$f_2(n) = \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2} = f_1(n) - f_1(n+1)$

and so on, so that in general (since ${}^kC_j = {}^kC_{j-1} + {}^{k-1}C_j$)

$$f_k(n) = f_{k-1}(n) - f_{k-1}(n+1).$$

From this formula, it follows by induction on k that

$$f_k(n) = \frac{k!}{n(n+1)(n+2)\ldots(n+k)} = \frac{1}{{}^{n+k}C_k . n}.$$

From the original expression $f_k(n) = \frac{m}{l.c.m.\{D_0, D_1, \ldots, D_k\}}$ for *some* integer m

and so the integer ${}^{n+k}C_k.n$ divides $l.c.m.\{D_0, D_1, \ldots, D_k\}$. Hence to prove

${}^{n+k}C_k.n = l.c.m.\ D_0, D_1, \ldots, D_k$ we need only show that D_j divides ${}^{n+k}C_k.n$

for every j. This is equivalent to showing $({}^{n+k}C_k.n)\,({}^kC_j \cdot \frac{1}{n+j})$ is an integer for

each j. But $(n.{}^{n+k}C_k)({}^kC_j . \frac{1}{n+j}) = \frac{n.(n+k)!\ k!}{(n+j)\ k!\ n!\ (k-j)!} =$

$= \frac{(n+j)-1)!}{(n-1)!\ j!} \cdot \frac{(n+k)!}{(n+j)!\ (k-j)!} = {}^{n+j-1}C_j . {}^{n+k}C_{k-j}$, which is clearly an integer.

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Little Boy Blue

Two little boys jumped into a swimming pool. "Boy it's cold," said the first boy, "the temperature of the water must be zero."

"Huh," said the other, "I'll bet it's twice as cold as that."

SUM WIZARDS

Are you very good at Mental Arithmetic? Can you do sums in your head? I don't know how good you are, but Zarah Colburn was extremely clever. He could multiply 8,476 by 2,549 in his head, and give the right answer in a few seconds. He was able to raise 8 to the 16th power – that is 8 x 8 x 8 x . . . sixteen times. He gave the answer, 281,474,976,710,656 in a couple of seconds. Oh, I forgot to tell you. He was only eight years old at the time. He was born in the USA in 1804. By his sixth birthday, he was travelling throughout the States, showing off his extraordinary powers. In 1812, his father took him to London, where he was asked those questions. He was able to say the square roots and cube roots of large numbers instantaneously. For example, the square root of 106,929 (which is 327) and the cube root of 268,336,125 (which is 645). He worked best when the answers were whole numbers. More impressive was his ability to state the factors of large numbers. Asked for the factors of 247,483 he replied 941 and 263; asked for the factors of 171,395 he gave 5, 7, 59 and 83; asked for the factors of 36,083 he said there were none. It must be added that he had difficulty handling numbers exceeding a million. This earlier success did not continue. His ability to calculate waned as he got older, and after a not very successful adult life, he died at the age of 36.

Even better at speed calculations was an English lad, George Parker Bidder (1806-1878). In 1818 he was pitted against Colburn, and Bidder proved to be the abler calculator. Here are some of the questions he answered, at about the age of 10.

1. Find the interest on £11,111 for 11,111 days at 5% per year.
   Answer:– £16,911/11/- given in about 1 minute.
2. If a coach wheel is 5'10" in circumference, how many times will it revolve in running 800,000,000 miles.
   Answer:– 724,114,285,704 times, with 20" remaining, given in about one minute.
3. If the pendulum of a clock vibrates the distance of 9¾" in a second, how many inches will it vibrate in 7 years 14 days 2 hours 1 minute 56 seconds, each year containing 365 days 5 hours 48 minutes 55 seconds.
   Answer:– 2,165,625,744¾ inches – given in less than a minute.

Unlike Colburn, Bidder retained his great mental ability and after graduating from Edinburgh University, became a Civil Engineer of some distinction.

Bidder's immediate family also showed signs of having extraordinary memories. One brother knew the Bible by heart and could name the chapter and verse of any text quoted to him. His son could multiply, in his head, two 15 digit numbers. But he could not match his father in speed and accuracy.

Just before we leave Bidder, let me quote an example of his phenomenal memory. In 1816, a number was read to him backwards – he at once gave it back in its normal form. An hour later he was asked if he remembered it: he immediately repeated it correctly. The number was –

2,563,721,987,653,461,598,746,231,905,607,541,128,975,231.

P. Rafferty

[Mr Rafferty is the Mathematics master at Wagga Wagga High School. – Ed.]

A Brain-teaser (submitted by Glenn Reeves, 1st Form, Newington (1974))

Alf, Bill, Charlie and Dick were very involved in the game of cricket on, as well as off work. It was only the other day I saw them chatting about their most recent game against Melbourne at the S.C.G.; arguing as usual. I wasn't really sure about which position each of the men held, until I was told that one was an umpire, who of course always told the truth, one a wicket-keeper, who also always told the truth, one a leg-break bowler, who always lied, and the fourth was the opening bat, who told alternate true and false, or false and true. These four men had one flat each, and all four flats were somewhere between flats 10 to 55. They spoke as follows:

ALF:
1. My flat is a multiple of 4.
2. I am the leg-break bowler.
3. Dick is the wicket-keeper.
4. The wicket-keeper's flat number is not prime.

BILL:
1. Charlie's fourth remark is false.
2. Alf's flat number is no greater than mine.
3. My flat number is not a multiple of 12.
4. Charlie's flat number is not a multiple of 7.

CHARLIE:
1. I am not the opening bat.
2. My flat number is 28.
3. The wicket-keeper's first name begins with 'R'.
4. The umpire's flat number is even.
5. The leg-break bowler's flat number is greater than 40.

DICK:
1. My flat number is not a multiple of 4.
2. I am not the umpire.
3. Bill is the leg-break bowler.
4. My flat number is greater than 52.

Who did what and which flat number does each person occupy? See answer below.

Answer to Brain-teaser

Alf was the opening bat, and lived in flat no. 52; Bill was the leg-break bowler, and lived in flat no. 48; Charlie was the umpire, and lived in flat no. 28; Dick was the wicket-keeper, and lived in flat no. 53.