PERFECT NUMBERS

A perfect number is a number which is equal to the sum of all its factors except itself. The lowest of such numbers is 6, whose factors (1, 2 and 3) total 6. Only 12 of these numbers are known. The first four are 6, 28, 496 and 8128; the fifth is 33 550 336 and the others are much greater. It must be noted that all of these known numbers end either with the digit 6 or the two digits 28; and so far no odd perfect numbers have been discovered.

Euclid proved that any number of the form $2^n\ (2^{n+1}-1)$ is a perfect number when the factor $(2^{n+1}-1)$ is a prime number. Thus 496 can be factorized to $2^4\ (2^5-1)$ or 16 x 31, and since 31 is prime, then 496 is perfect. It will be observed that the known perfect numbers become much less frequent as numbers grow larger. This is because the primes on which they are based become more rare.

There is a way of building up perfect numbers without having direct regard to Euclid's formula. The factors of 6 are 1, 2 and 3; and these are consecutive positive integers (called an arithmetic progression). This provides a clue that the other perfect numbers may also prove to be equivalent to the sums of arithmetic progressions. This is, in fact, the case; and each of these progressions is inter-related to the others. The first five perfect numbers can now be shown:

6	=	Sum of progression 1 to 3	inclusive
28	=	Sum of progression 1 to 7	inclusive
496	=	Sum of progression 1 to 31	inclusive
8 128	=	Sum of progression 1 to 127	inclusive
33 550 336	=	Sum of progression 1 to 8 191	inclusive

The last term is, in every case, a prime (being of the same form as in Euclid's formula), and the only other factors of the perfect numbers are 1 and varying powers of the number 2.

The relationship between each progression is as follows. Once the last term l_1 of an earlier progression is known, the last term of the next progression l_2 is found by multiplying l_1 by 2 and adding 1. So $l_2 = 2l_1 + 1$; but only provided the resulting l_2 is prime. For example, as between the two numbers 6 and 28, the last terms of their respective progressions are 3 and 7 (that is, $l_1 = 3$, $l_2 = 7$; and so $l_2 = 2l_1 + 1$). If however, the resulting l_2 is not a prime then the procedure of doubling and adding 1 (at each stage) is continued until a prime does arise. This prime, when reached, will then be the last term of the next perfect number's progression.

This explains the gap between the fourth and fifth perfect numbers. The last term of the progression for the fourth number is 127. The successive doubling up and addition of unity at each stage give the following:

(Fourth number)	Last term of A.P. =	127 Prime
	double and add 1 =	255 Not Prime
	double and add 1 =	511 Not Prime
	double and add 1 =	1023 Not Prime
	double and add 1 =	2047 Not Prime
	double and add 1 =	4095 Not Prime
	double and add 1 =	8191 Prime

The last term of each progression also bears a direct relationship to its perfect number other than as a mere factor, and it is instructive to compare the structure of a perfect number (regarded as the sum of an arithmetic progression) with its factorial composition.

The perfect number 496 is the sum of the series 1 to 31, and its various terms may be regrouped as follows:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
31	30	29	28	27	26	25	24	23	22	21	20	19	18	17.	

Each of the first 15 groups totals 32, leaving the number 16 unpaired. The total of all the terms is therefore $15 \times 32 + 16$ or $15\frac{1}{2} \times 32$.

On the other hand, the factors of 496 are 1, 2, 4, 8, 16, 31, 62, 124, 248, and these can be re-grouped thus:

$$
\begin{aligned}
1 + 31 &= 32 = 1 \times 32 \\
2 + 62 &= 64 = 2 \times 32 \\
4 + 124 &= 128 = 4 \times 32 \\
8 + 248 &= 256 = 8 \times 32 \\
16 &= 16 = \tfrac{1}{2} \times 32 \\
\text{Total} & \quad 15\tfrac{1}{2} \times 32
\end{aligned}
$$

In each case the number 496 is shown as being $15\frac{1}{2} \times 32$ or $\frac{31 \times 32}{2}$.

The fact that these known perfect numbers are equal to the sums of arithmetic progressions of the forms 1, 2, 3, 4, . . . means that they must also be triangular numbers. However, they are particular triangular numbers and it does not follow that all triangular numbers are perfect. (For example, 15 is a triangular number but not a perfect number.)

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