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MATHEMATICS AND THE MAJOR SCALE

A talk given to high school level 1 students on June 20, 1974 by J. McMullen of Sydney University.

Everybody knows that, because of Pythagoras' theorem, the diagonal of a unit square must have irrational length. Conservatives nonetheless held up the legalization of irrational numbers for a long time. Not so many people know, however, that, while the natural scales used for music up to the fifteenth century were based on rational numbers, today's scales are based on irrational numbers, and for a very good reason.

The first music theoreticians were, oddly enough, the followers of Pythagoras. They noticed that a string stretched tightly between two fixed points can vibrate in a number of ways, shown in figure 1. Thus it can vibrate in its own "fundamental mode" (a), or as if it were two strings each of half the original length (b), or three strings each of one third the length (c), or in fact as n strings each of $\frac{1}{n}$ times the original length. The notes so produced are called the "harmonic series" of the string (the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$ is still called the harmonic sequence in mathematics).

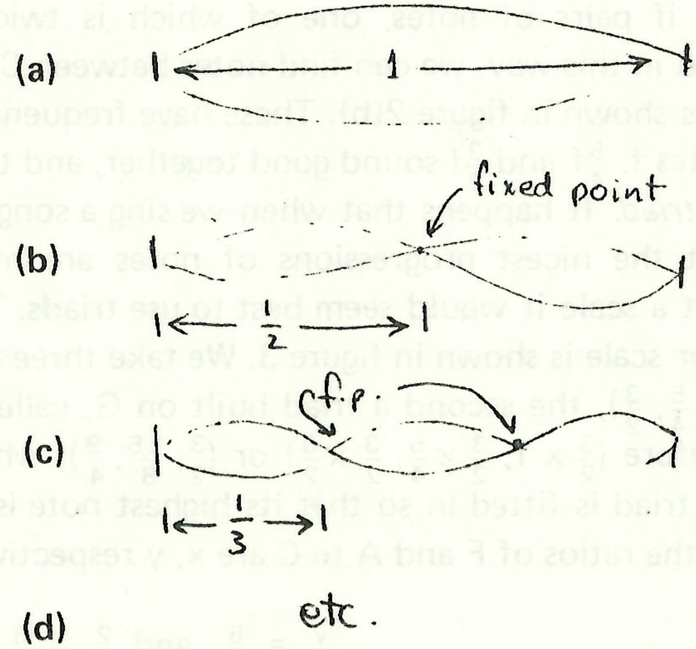

Figure 1

Now, usually a string vibrates in all these modes *at the same time,* and we may deduce that all the notes so produced sound good *together.* Since the first, say, six harmonics will be by far the loudest, combinations of these will be the best sounding ones of all. Thus we consider a string divided in the ratios $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{6}$. It is also convenient to talk of the frequency of the note rather than the length of the string: these are in inverse proportion, and so if the fundamental has a frequency f, we are considering notes of frequencies 2f, 3f, 4f, 5f and 6f. Now we can construct, say, a major scale.

First of all, doubling the frequency produces an interval of an *octave* (say, from Middle C on a piano to the next C) and these notes are so alike that, of course, they are given the same name. Similarly, the notes 2f, 4f and the notes 3f and 6f, sound pretty much the same, and we use the same letter for these. Thus in figure 2(a) we have called the notes f, 2f and 4f C, C′ and C″ respectively, and the notes 3f and 6f we call G′, G″. The note 5f we call E″.

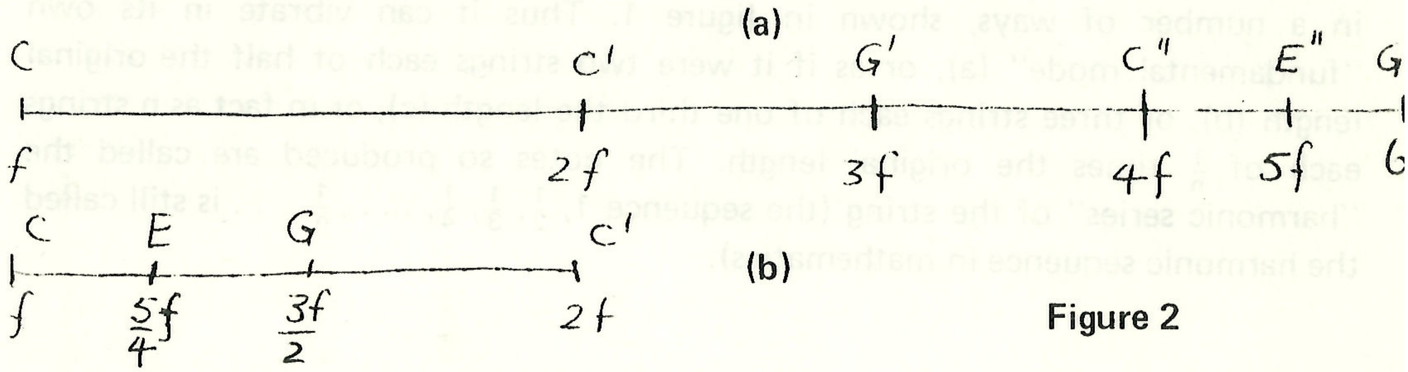

Figure 2

Now, if pairs of notes, one of which is twice the other in frequency are identified in this way, we can find notes between C and C′ that should be called G and E, as shown in figure 2(b). These have frequencies $\frac{3}{2}$f and $\frac{5}{4}$f respectively. The three notes f, $\frac{5}{4}$f and $\frac{3}{2}$f sound good together, and the set of ratios $1, \frac{5}{4}, \frac{3}{2}$ is called a *major triad.* It happens that when we sing a song, we sing a succession of notes, and that the nicest progressions of notes are members of a *triad.* Hence to construct a scale it would seem best to use triads. The way this has been done for the major scale is shown in figure 3. We take three triads, the first our old (C,E,G) triad $(1, \frac{5}{4}, \frac{3}{2})$, the second a triad built on G, called (G,B,D′). The ratios for this are therefore $(\frac{3}{2} \times 1, \frac{3}{2} \times \frac{5}{4}, \frac{3}{2} \times \frac{3}{2})$ or $(\frac{3}{2}, \frac{15}{8}, \frac{9}{4})$, when compared with the note C. The last triad is fitted in so that its highest note is C′. Let us call it (F,A,C′), and suppose the ratios of F and A to C are x, y respectively. Then

$$\frac{y}{x} = \frac{5}{4}, \text{ and } \frac{2}{x} = \frac{3}{2}$$

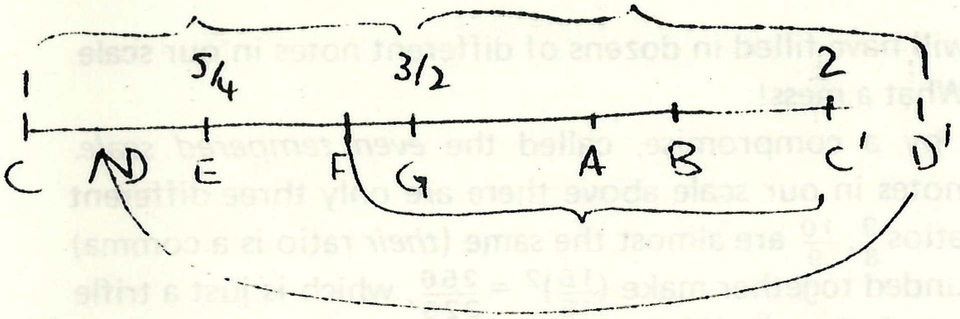

Figure 3

so that $x = \frac{4}{3}$ and $y = \frac{5}{3}$. Finally, we shift D′ down an octave to get a note D which is $\frac{1}{2} \times \frac{9}{4}$ or $\frac{9}{8}$ when compared with C. Thus we have a major scale on C:

C	D	E	F	G	A	B	C′
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

The reason for using these letters is now seen to be the order of these fractions by size.

This is all very well, but what happens when we want to sing or play in another key? What we need is a scale starting, say, on F, and using the same ratios. Thus, as F has frequency ratio $\frac{4}{3}$ compared with C, our F major scale needs the notes:

$$\frac{4}{3} \times 1,\ \frac{4}{3} \times \frac{9}{8},\ \frac{4}{3} \times \frac{5}{4},\ \frac{4}{3} \times \frac{4}{3},\ \frac{4}{3} \times \frac{3}{2},\ \frac{4}{3} \times \frac{5}{3},\ \frac{4}{3} \times \frac{15}{8},\ \frac{4}{3} \times 2$$

or when simplified

$$\frac{4}{3},\ \frac{3}{2},\ \frac{5}{3},\ \frac{16}{9},\ 2,\ \frac{20}{9},\ \frac{5}{2},\ \frac{8}{3}$$

where if we use our old notes F, G, A, B, C, D′, E′, F′, we get

$$\frac{4}{3},\ \frac{3}{2},\ \frac{5}{3},\ \frac{15}{8},\ 2,\ \frac{9}{4},\ \frac{5}{2},\ \frac{8}{3}$$

(the last three are got by multiplying $\frac{9}{8}$, $\frac{5}{4}$, $\frac{4}{3}$ by 2). Thus two notes are wrong, B and D′. B is too high by a ratio

$$\left(\frac{15}{8}\right) \ / \ \left(\frac{16}{9}\right) \ = \ \frac{135}{128}$$

which is almost as big as the ratio of F to E or of C to B (compare them) and so we need a new note B^b (B flat). However, D′ is too high by a ratio of $\frac{81}{80}$ (a "comma") which is quite small! It turns out that after we add notes for the scales on G, A, B, D, and E, we will have filled in dozens of different notes in our scale, some only a comma apart. What a mess!

This problem is solved by a compromise, called the *even tempered scale.* Between pairs of adjacent notes in our scale above there are only three different ratios: $\frac{9}{8}$, $\frac{10}{9}$, and $\frac{16}{15}$. The ratios $\frac{9}{8}$, $\frac{10}{9}$ are almost the same (*their* ratio is a comma) while two $\frac{16}{15}$ ratios compounded together make $(\frac{16}{15})^2 = \frac{256}{225}$, which is just a trifle bigger (by less than a comma) than $\frac{9}{8}$. What we do is adjust all the notes by dividing the octave (C,C′) into twelve equal parts, so that the big intervals (C,D), (D,E), (F,G) and (G,A) are each made up of two small intervals the same size as (E,F) and (B,C). Suppose that the first of these division notes has ratio x compared to C. Then since twelve of these make an octave, x must satisfy the equation

$$x^{12} = 2$$

so that $x = \sqrt[12]{2}$. Our scale is now

C	D	E	F	G	A	B	C′
1	x^2	x^4	x^5	x^7	x^9	x^{11}	x^{12}

and we put in "black" notes to correspond with the other values x, x^3, x^6, x^8, x^{10}. Now we can start at any note and all major scales will have the same sound, although none will be quite right to the ear (work out the ratios and compare them with the "natural" scale).

Pythagoras would never have approved of this scale. $\sqrt[12]{2}$ indeed! But J.S. Bach wrote forty-eight preludes and fugues in all possible keys to promote the idea. Perhaps owing to this brilliant advertising campaign, we use this scale, more or less, today.

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Twin Primes

Since our article "Of Prime Interest" in Vol. 10 No. 1, a reader has found in the Journal of Recreational Mathematics a much larger pair of twin primes than the ones we gave. They are $9 \times 2^{211} - 1$ and $9 \times 2^{211} + 1$, and require 65 digits when written to base 10. A still larger pair can be found in Volume 26 of the journal Mathematics of Computation. They are

$$76 \times 3^{139} - 1 \quad \text{and} \quad 76 \times 3^{139} + 1$$

and these require 68 digits when written to base 10. Who wants to try for a larger pair?