SCHOOL MATHEMATICS COMPETITION 1975

Junior Division

Question 1: The sum of two positive integers is 10 000 000 000. The digits of the two numbers are the same but not in the same order.
Prove that both numbers are divisible by 5.

Answer: Suppose the numbers are A and B, and that they end in the digits a, b respectively. Since A + B ends on 0, either a + b = 0 or a + b = 10. Suppose $a \neq b$. Then $a + b \neq 0$, so a + b = 10, so 1 is carried. Since A + B = 10,000,000,000, the two digits in each column to the left of the units column must add to 9. So if the digit d occurs in any column other than the units column in one of A and B, then the digit $\overline{d} = 9-d$ occurs in the same position in the other number. Note that $\overline{d} \neq d$. Now A contains the digit a, so B contains the digit a, but not in the units column, since $b \neq a$. So A contains $\overline{a}$. So B contains $\overline{a}$, but not in the units column, since $b = 10-a \neq \overline{a}$. So far, we know

$$A = \ldots\overline{a}\ldots\ldots\ldots\ldots a$$
$$B = \ldots a\ldots\overline{a}\ldots\ldots\, b$$

So A contains $\overline{\overline{a}} = 9-\overline{a} = 9-(9-a) = a$ once again. So B contains a once again. And we can see that both A and B would have to contain the digit a infinitely often, which is impossible.

So a = b. Since a + b = 0 or 10, either a = b = 0 or a = b = 5. In either case, both A and B are divisible by 5.

Note that we didn't use the fact that the digits of A and B are not in the same order.

A different method enables one to prove that in fact both A and B are divisible by 10, but we haven't room enough here. ☆

Question 2: (a) In a class of 12 students (4,6,5) means that a student was fourth in the first exam paper and sixth in the second paper; and that, when the marks were added, he or she came fifth in the whole examination.

Which of the following are possible results and which ones are impossible?

(i) (4,5,7);   (ii) (10,11,12);   (iii) (3,5,8);
(iv) (1,1,2);   (v) (1,12,1);   (vi) (8,9,4).

(b): In a class of n students find conditions on a, b and c in order that (a,b,c) be an impossible result.

Assume that there are no tied marks in any of the three lists.

Answer: **(a):** (4,5,7), (10,11,12), and (1,12,1) are possible, (3,5,8), (1,1,2), and (8,9,4) are impossible.

(b): (a,b,c) is impossible if $c < a + b - n$ or $c > a + b - 1$.

If a student comes a'th in the first exam and b'th in the second then (a–1) students beat him in the first, (b–1) in the second. If $(a-1) + (b-1) > (n-1)$, then at least $(a-1) + (b-1) - (n-1) = (a + b - n - 1)$ students beat him in both, so he can't be better than (a + b–n)'th overall, i.e. $c < a + b - n$ is impossible.

Similarly, he has beaten (n–a) in the first exam, (n–b) in the second. If $(n-a) + (n-b) > (n-1)$, then he beat at least $(n-a) + (n-b) - (n-1) = (n-a-b+1)$ in both, so he can't be worse than (a + b–1)'th overall, i.e. $c > a + b - 1$ is impossible.

If $a + b - n \leqslant c \leqslant a + b - 1$ (and, of course, $1 \leqslant c \leqslant n$) then (a,b,c) is a possible result, but we will not show it here. ✫

Question 3. **(a):** Prove that the product of three consecutive positive integers is never a cube.

(b): $8 = 2^3$ and $9 = 3^2$ are the only known examples of two consecutive integers that are powers. No one has yet been able to show that there are no other pairs of such integers.

Prove the much easier statement that there are no four consecutive integers such that all four are powers of some integers.

Note: "power" means higher than first power.

Answer: **(a):** Let the three positive integers by x–1, x, x + 1, and suppose $(x-1)x(x+1) = y^3$.

Clearly, $(x-1)^3 < (x-1)x(x+1) < (x+1)^3$,

so $(x-1)^3 < y^3 < (x+1)^3$,

so $(x-1) < y < (x+1)$.

So if y were an integer, we would have y = x. But $(x-1)x(x+1) \neq x^3$, so $(x-1)x(x+1)$ cannot be the cube of an integer.

(b): Of any four consecutive integers, one must leave a remainder 2 when divided by 4. This number is even. But it cannot be a power of an integer, because that integer would have to be even, and any power of an even number leaves a remainder 0 when divided by 4, not a remainder 2. ☆

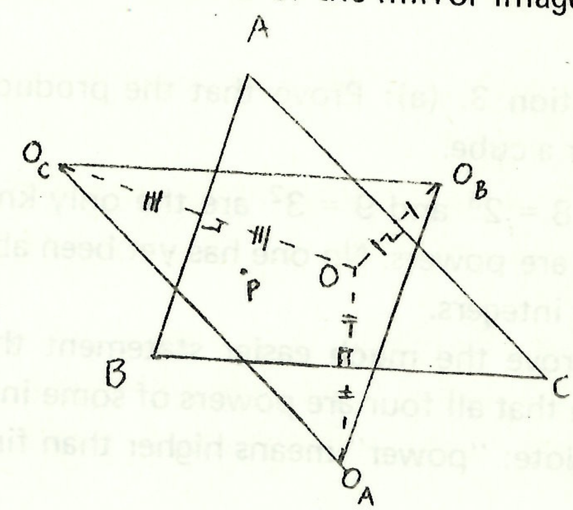
Question 4: Given a triangle ABC with acute angles, show how to find a point P such that the four circumscribed circles around the four triangles ABC, ABP, BCP, CAP have equal radii but do not coincide (that is, P should not lie on the circumscribed circle of ABC). If you have constructed such a point P, prove that it has the required property.

Discuss the case when ABC is not acute angled.

Answer: Let the centres of the four circumscribed circles be O, O_C, O_A, O_B respectively. Then

$$OA^* = OB^* = OC^* = O_C A^* = O_C B^* = O_C P^* = O_A B^* = O_A C^* = O_A P^* = O_B C^* = \\ = O_B A^* = O_B P^*.$$

Since $OA^* = O_C A^*$ and $OB^* = O_C B^*$, O_C must either be O or the mirror image of O in the line AB. If $O_C = O$, then $OP^* = OA^* = OB^* = OC^*$ and so P is on the circumscribed circle of ABC (not allowed by the question). Thus O_C is the mirror image of O in AB.



Similarly, O_A is the mirror image of O in BC and O_B is the mirror image of O in CA.

Now, since $O_C P^* = O_A P^* = O_B P^*$, P is the circumcentre of the triangle $O_C O_A O_B$.

Thus P is constructed by first constructing O_A, O_B, O_C; and then by finding the circumcentre of $\Delta O_A O_B O_C$. This construction will always work provided O is not on one of the sides of the triangle, i.e. provided the triangle is not right-angled. ☆

[Parabola readers might like to find some interesting properties of the point P and the triangles. – Editor]

Question 5: Each face of a cube is coloured by a different colour, say, red, purple, green, blue, yellow and white. In how many different ways can this be done? Two colourings are the same if the two cubes can be placed parallel to each other so that they show the same colours on corresponding faces (top, bottom, front, back, left, right).

Answer: There are 30 different cubes. Every cube has a red face. Place it at the front. There are then five possible colours for the back. Choose one, say purple. Then one of the remaining faces is white. Place it at the bottom. There are then three possible colours for the top. Choose one, say yellow. There are then two possible colours for the left side. Choose one, say green. Then the right side must be blue.

The number of possible cubes is $5 \times 3 \times 2 = 30$. ✯

Senior Division

Question 1: Let x be a positive real number. Which of the following statements is (or are) true:

(a) If both x^7 and x^{12} are rational numbers then x is rational.

(b) If both x^9 and x^{12} are rational then x is rational.

If the statement is true, prove it, if false give a counterexample. Can you generalize to x^m and x^n for arbitrary integer exponents m, n?

Answer: (a) is true, (b) is false.

If x^7 and x^{12} are rational then so are $(x^7)^5$ and $(x^{12})^3$ i.e. x^{35} and x^{36} are rational. So x is rational.

If x^9 and x^{12} are rational, then x^3 is rational, but x need not be rational. For example, if $x = \sqrt[3]{2}$, x is irrational, but $x^9 = 8$ and $x^{12} = 16$ are rational.

If x^m and x^n are rational, and the g.c.d. of m and n is 1, then x is rational. For we can find integers a and b such that $am + bn = 1$. x^m is rational, so x^{am} is rational. x^n is rational, so x^{bn} is rational. So $x^{am+bn} = x$ is rational.

If the g.c.d. of m and n is $d > 1$, and if $x = \sqrt[d]{2}$, x is irrational, but x^m and x^n are rational. ✯

Question 2: (a): Prove that the product of three consecutive positive integers is never a perfect power of an integer (higher than the first).

(b): Prove that the product of k consecutive positive integers ($k > 1$) is never a k-th power.

Answer: (a): Let the three positive integers be $x-1$, x, $x+1$, and suppose $(x-1)x(x+1) = y^k$. Then $x(x^2-1) = y^k$. Now x and x^2-1 are relatively prime, so each must be a k'th power. Let $x = a^k$, $x^2-1 = b^k$. Then $a^{2k}-1 = b^k$, or $c^k - b^k = 1$, where $c = a^2$. But it is impossible for two k'th powers to differ by 1 (except for $c = 1, b = 0$ or, if k odd, $c = 0, b = -1$, neither of which hold here since $x \geqslant 2$).

(b): Let the k positive integers be $x+1, x+2, \ldots, x+k$, and suppose $(x+1)(x+2)\ldots(x+k) = y^k$ (where y is positive).

Now $(x+1)^k < (x+1)(x+2)\ldots(x+k) < (x+k)^k$,
so $(x+1)^k < y^k < (x+k)^k$,
so $x+1 < y < x+k$.

If y is an integer, $y = x + r$ for some r with $2 \leqslant r \leqslant k-1$.
So $(x+1)\ldots(x+r)\ldots(x+k) = y^k = (x+r)^k$,
so $(x+1)\ldots(x+r-1)(x+r+1)\ldots(x+k) = (x+r)^{k-1}$.
Let p be any prime which divides $x+r+1$. Then p divides the left-hand-side. But p does not divide $x+r$, so p does not divide the right-hand-side, a contradiction. So $(x+1)\ldots(x+k)$ is not the k'th power of an integer.

Question 3: (a): Prove that in a triangle the lengths of altitude, angular bisector and median drawn from the same vertex follow each other in the same order.

(b): Assume further the triangle to be acute angled. Draw the circumscribed circle. Extend each of the above lines to meet the circle. Prove that the new lengths are still in the same order.

(c): Consider the extended length of the angle bisector. Prove that it is larger than the arithmetic mean of the two sides comprising the angle which has been bisected.

Answer: Let A be the vertex, BC the opposite side, M the midpoint of BC and AX the bisector of the angle A (see Figure 1 opposite), where we may assume AB* > AC*. (If AB* = AC*, the triangle is isosceles, and the altitude, angle bisector and median all coincide.)

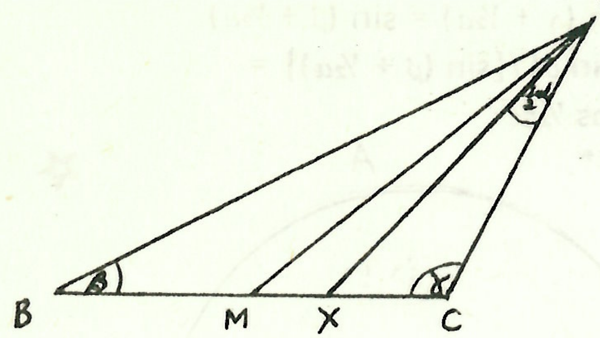


Figure 1

(a): In ΔACX, $CX^*/(\sin \frac{1}{2}\alpha) = AC^*/(\sin \angle CXA)$. In ΔABX, $BX^*/(\sin \frac{1}{2}\alpha) = AB^*/(\sin \angle BXA) = AB^*/(\sin \angle CXA)$ since $\angle BXA + \angle CXA = 180°$. Hence $CX^*/BX^* = AC^*/AB^* < 1$ and so $CX^* < BX^*$. Thus $CX^* < \frac{1}{2}CB^* = CM^*$, whence X is between M and C. Now, since $AB^* > AC^*$, $\gamma > \beta$ (the larger angle being opposite the longer side AB). Thus $\angle AXM = \frac{1}{2}\alpha + \gamma > \frac{1}{2}\alpha + \beta = \angle BAM + \beta = \angle AMX$, and so $AM^* > AX^*$.

Finally if D is the foot of the altitude through A, then $AX^* > AD^*$ since AX is the hypotenuse of the right-angled triangle AXD.

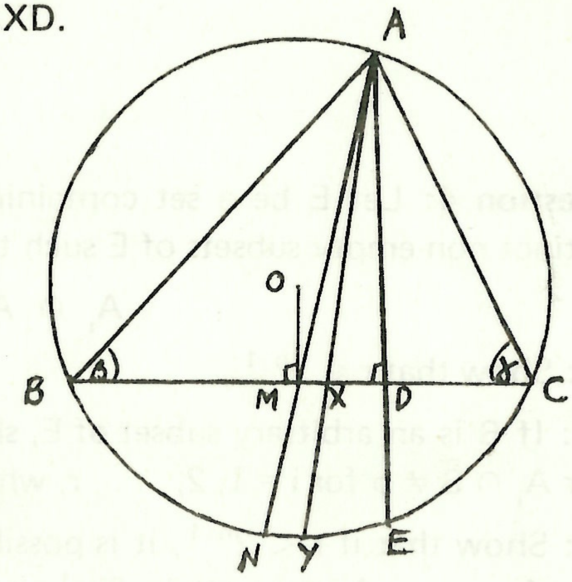


Figure 2

(b): Suppose the lines AM, AX, AD above meet the circumcircle at N, Y, E respectively. The centre O of the circumcircle lies on the perpendicular bisector OM of BC and, since $\angle BAC$ is acute, O is on the same side of BC as A is. (The arc BC subtends at O an angle equal to $2\alpha < 180°$, so this arc is less than half the circumference.) Also, since $\gamma < 90°$, D lies between B and C, and $\angle CAD = 90° - \gamma < 90° - \beta = \angle BAD$. Thus $\angle CAD < \frac{1}{2}\alpha$ and so D lies between X and C. It is now clear that the chords AN, AY and AE are in the order shown in Figure 2, with AN closest to O. Since the length of a chord is less the further it is from the centre, we deduce that $AN^* > AY^* > AE^*$.

(c): $\angle YBC = \angle YAC = \frac{1}{2}\alpha$, $\angle YCB = \angle YAB = \frac{1}{2}\alpha$, etc. (see Figure 3.)
In ΔABY, $AB^*/(\sin \gamma) = AY^*/[\sin(\beta + \frac{1}{2}\alpha)]$, so $AB^*/AY^* = \sin\gamma/[\sin(\beta + \frac{1}{2}\alpha)]$.
In ΔACY, $AC^*/(\sin \beta) = AY^*/[\sin(\gamma + \frac{1}{2}\alpha)]$, so $AC^*/AY^* = \sin\beta/[\sin(\gamma + \frac{1}{2}\alpha)]$.
Therefore $(AB^* + AC^*)/AY^* = \sin\gamma/[\sin(\beta + \frac{1}{2}\alpha)] + \sin\beta/[\sin(\gamma + \frac{1}{2}\alpha)]$.

Now $\alpha + \beta + \gamma = 180^\circ$, so $\sin\gamma = \sin(\alpha + \beta)$, $\sin(\gamma + \frac{1}{2}\alpha) = \sin(\beta + \frac{1}{2}\alpha)$.
Therefore $(AB^* + AC^*)/AY^* = [\sin(\alpha + \beta) + \sin\beta]/[\sin(\beta + \frac{1}{2}\alpha)] =$
$[2\sin(\beta + \frac{1}{2}\alpha)\cos(\frac{1}{2}\alpha)]/[\sin(\beta + \frac{1}{2}\alpha)] = 2\cos\frac{1}{2}\alpha$.
Therefore $\frac{1}{2}(AB^* + AC^*) = AY^*\cos\frac{1}{2}\alpha < AY^*$.

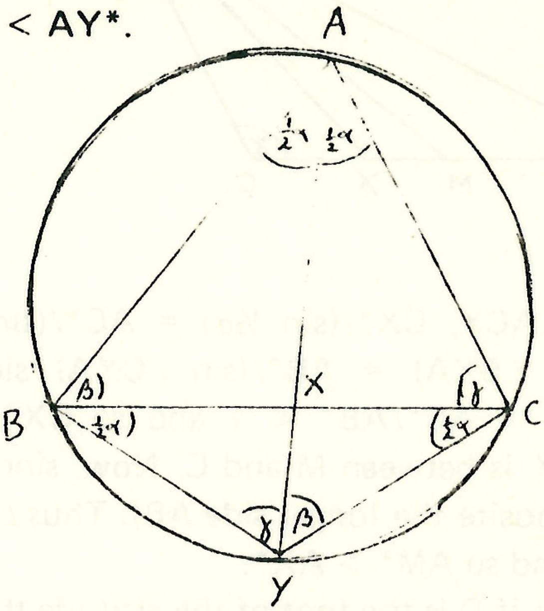


Figure 3

Question 4: Let E be a set containing n elements and let $A_1, A_2, \ldots, A_r$ be distinct non-empty subsets of E such that

$$A_i \cap A_j \neq \phi, \quad i \neq j. \qquad (1)$$

(a): Show that $r \leqslant 2^{n-1}$.

(b): If B is an arbitrary subset of E, show that either $A_i \cap B \neq \phi$ for $i = 1, 2, \ldots, r$ or $A_i \cap \overline{B} \neq \phi$ for $i = 1, 2, \ldots, r$, where $\overline{B}$ is the complement of B in E.

(c): Show that if $r < 2^{n-1}$, it is possible to extend any given collection of subsets $A_1, A_2, \ldots, A_r$ to contain 2^{n-1} subsets with property (1). (That is, there are subsets $A_{r+1}, \ldots, A_{2^{n-1}}$ such that (1) is true for $1 \leqslant i < j \leqslant 2^{n-1}$.)

ϕ denotes the empty subset. Recall that the number of all distinct subsets of a set of n elements (including ϕ and the set itself) is 2^n.

Answer: (a): If we have a collection of r subsets of E, where $r > 2^{n-1}$, then the collection must contain a pair A, $\overline{A}$, where $\overline{A}$ is the complement of A. Since $A \cap \overline{A} = \phi$, the collection does not have property (1).

(b): Suppose we have a collection $A_1, \ldots, A_r$ of subsets of E with property (1), and let B be any subset of E. Suppose both statements are false. Then for some i,

$A_i \cap B = \phi$, and for some j, $A_j \cap \overline{B} = \phi$. These imply that $A_i \subset \overline{B}$ and $A_j \subset B$. Since $B \cap \overline{B} = \phi$, $A_i \cap A_j = \phi$, a contradiction. So either $A_1 \cap B, \ldots, A_r \cap B$ are all non-empty, or $A_1 \cap \overline{B}, \ldots, A_r \cap \overline{B}$ are all non-empty.

(c): Suppose $A_1, \ldots, A_r$ is a collection with property (1), and suppose $r < 2^{n-1}$. Then there is a subset B of E such that neither B nor $\overline{B}$ is one of the A's. Now by (b), one of B, $\overline{B}$ (suppose B, for the sake of argument) has the property that $A_1 \cap B, \ldots, A_r \cap B$ are all non-empty. Add this subset to the collection, calling it A_{r+1}. In this way the original collection can be extended one subset at a time to a collection of 2^{n-1} subsets with property (1).

Question 5: Let B(n) denote the number of partitions of n into powers of 2, e.g. B(6) = 6 since

$$6 = 1+1+1+1+1+1 = 1+1+1+1+2 = 1+1+2+2$$
$$= 2+2+2 = 1+1+4 = 2+4.$$

Prove that

(a) $B(2n+1) = B(2n)$

(b) $B(2n) = B(2n-1) + B(n)$

(c) B(n) is even for $n > 1$

(d) B(2n) is divisible by 4 if and only if $n = 2^r m$ where both r amd m are odd.

Answer: (a): Any partition of 2n + 1 into powers of 2 must contain at least one 1. If we delete one 1 from each such partition, we obtain the various partitions of 2n into powers of 2, and this process is reversible. So $B(2n + 1) = B(2n)$.

(b): A partition of 2n into powers of 2 either contains a 1 or it doesn't. In the first case, if we delete a 1, we obtain a partition of 2n−1 into powers of 2. In the second, if we divide all the parts by 2, we obtain a partition of n into powers of 2. Moreover, both processes are reversible. So $B(2n) = B(2n-1) + B(n)$.

(c): $B(2) = 2$ is even. Suppose $B(2), \ldots, B(k)$ are even, and consider $n = k+1$. If $k+1$ is odd, $B(k+1) = B(k)$ is even. If $k+1$ is even, $B(k+1) = B(k) + B(\frac{1}{2}(k+1))$, and since $\frac{1}{2}(k+1) \geqslant 2$, B(k) and $B(\frac{1}{2}(k+1))$ are both even, so $B(k+1)$ is even. So B(n) is even for $n > 1$, by induction.

(d): First note that any number n can be written in the form $2^r m$ with m odd. Then r is a function of n, which we shall denote by r(n). Second, B(2n) is even, so if we divide B(2n) by 4, the remainder is 0 or 2. In the first case write $B(2n) \equiv 0$, in the second $B(2n) \equiv 2$.

We have to show that if r(n) is odd, $B(2n) \equiv 0$, while if r(n) is even, $B(2n) \equiv 2$. We prove this proposition by induction.

r(1) = 0 is even, $B(2) = 2 \equiv 2$, so the proposition is true for n = 1.

Suppose the proposition true for n = 1, 2, . . . , 2k−1.

Consider n = 2k.

$$B(2n) = B(4k) = B(4k-1) + B(2k) \quad \text{by (b)}$$
$$= B(4k-2) + B(2k) \quad \text{by (a).}$$

Now r(2k−1) = 0, so, since the proposition is true for n = 2k−1, $B(4k-2) \equiv 2$. If r(n) is odd, i.e. r(2k) is odd, r(k) is even, so $B(2k) \equiv 2$, so $B(2n) \equiv 0$, while if r(n) is even, i.e. r(2k) is even, r(k) is odd, so $B(2k) \equiv 0$, so $B(2n) \equiv 2$. Thus the proposition is true for n = 2k.

Now consider n = 2k + 1.

$$B(2n) = B(4k+2) = B(4k+1) + B(2k+1) \quad \text{by (b)}$$
$$= B(4k) + B(2k) \quad \text{by (a).}$$

One of r(2k), r(k) is even, the other odd, so, since the proposition is true for n = 2k and n = k, one of B(4k), $B(2k) \equiv 2$, the other $\equiv 0$, so $B(2n) \equiv 2$, while r(n) = 0. So the proposition is true for n = 2k + 1.

This completes the proof. ✫

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A quickie for Bookworms

Submitted by Jim Pike

See if you can answer the following question in less than 10 seconds:

A "bookworm" eats his way from the first page of Vol. 1 to the last page of Vol. 2 of the encyclopaedia on your school library's shelf. If the covers of these books are ½ cm thick and the insides of these books are both 2 cm thick, how far did the worm eat?

A Metallic Cryptarithm

Submitted by Greg Clark, Form 2, St. Joseph's College

Replace each of the letters in the following sum by different digits to get a true result. As is usual, O represents an odd number and E represents an even number.

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  S T E E L  +
  B R A S S
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C O P P E R
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Answers on page 36