PROBLEM SECTION

Students are invited to submit solutions of one or more of these problems. Answers should bear the author's name, class and school. Model solutions, and the names of those who submit solutions before the publication of the next issue, will be published in Vol. 11 No. 3.

Although the problems are unclassified, the first 4 problems should require no more mathematical knowledge than is learnt in first or second form and the next 4 problems should require no more mathematical knowledge than is learnt in the first 4 forms of High School. Senior students are thus encouraged to attempt the later problems.

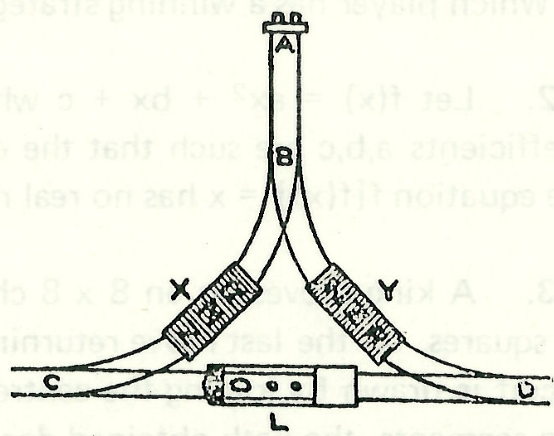
273. What is the smallest and largest possible number of Fridays that can occur on the 13th of a month in any calendar year (e.g. Friday 13th June is the only one in 1975).

274. A triangle has area 1 sq. cm and sides of length a cm, b cm, c cm where $a \geqslant b \geqslant c$. Prove that $b \geqslant \sqrt{2}$.

275. *(From Mathematical Digest, a New Zealand magazine)*

The diagram shows a straight railway line with two sidings. That part AB, common to both sidings, is long enough to contain either of the two wagons X and Y but not both at once. L, the locomotive, is too long to go on AB.

How can the positions of X and Y be interchanged? (The couplings can be connected or disconnected only while the locomotive and wagons are stationary.)



276. (198, 199, 200, 201, 202) is a set of consecutive positive integers whose sum is 1000. Find all such sets.

277. When $p = 3$, $p^3 + p^2 + 11p + 2 = 71$ is a prime number. Prove that 3 is the only prime number value of p for which $p^3 + p^2 + 11p + 2$ is also prime.

278. At a party the guests are lined up so that each person (with the exception of the two at the ends) is acquainted with exactly as many people to his right as to his left. Show that the first and last person have the same number of acquaintances.

279. On each side of a convex quadrilateral a circle is drawn having that side as diameter. Prove that every point inside the quadrilateral lies inside at least one of the 4 circles.

280. Find all solutions of the simultaneous equations:

$$y = x + \sqrt{(x + \sqrt{(x + \ldots + \sqrt{(x + \sqrt{y})) \ldots)}}}$$

$$x + y = 6$$

where there are 1975 square roots in the first equation.

281. Two people play the following game on an 8 x 8 chess board: A pawn is placed on the lower left corner square and moved alternately by the players to a neighbouring square either up, or to the right, or diagonally up and right. The game stops when the pawn reaches the upper right corner square, the player making the final move being the winner.

Which player has a winning strategy, and what is it?

282. Let $f(x) = ax^2 + bx + c$ where a,b,c are real numbers. Prove that if the coefficients a,b,c are such that the equation $f(x) = x$ has no real roots then also the equation $f[f(x)] = x$ has no real roots.

283. A king moves on an 8 x 8 chessboard so that in 64 moves it goes through all squares, on the last move returning to its original position. Furthermore, if the circuit is drawn by joining the centre points of consecutive positions with straight line segments, the path obtained does not cross itself. Prove that at least 28 of the moves have been either horizontal or vertical.

284. You are given 50 intervals on a line. Prove that at least one of the following statements about those intervals is true:

(a) There are 8 intervals all of which have at least one point in common.

(b) There are 8 intervals so that no two of them have a common point.

Solutions to Problems 261–272 (Vol. 11 No. 1)

261. In a right-angled triangle, the shortest side is a cm long, the longest side is c cm long and the other side is b cm. If a, b, c are all integers, when does $a^2 = b + c$?

Answer: By Pythagoras' Theorem, $a^2 = c^2 - b^2 = (c-b)(c+b)$. This equals $c + b$ if and only if $c - b = 1$ (i.e. when the hypotenuse is just one unit longer than the next longest side).

Thus

$$\begin{aligned} a^2 &= c^2 - b^2 \\ &= (b+1)^2 - b^2 \\ &= 2b + 1 \end{aligned}$$

a cannot be even and so $a = 2d + 1$ for some integer d.

So

$$\begin{aligned} 2b + 1 &= (2d+1)^2 \\ &= 4d^2 + 4d + 1 \end{aligned}$$

i.e. $b = 2d^2 + 2d$ and $c = b + 1 = 2d^2 + 2d + 1$.

262. On his birthday in 1975 John reaches an age equal to the sum of the digits in the year he was born. What year was that?

Answer: The sum of the digits of any date earlier than 1975 does not exceed 27 (which occurs for the year 1899). Hence John is at most 27 years old and was born no earlier than 1948. The largest sum of digits of a date between 1948 and 1975 is 25 (the year 1969), and the smallest such sum is 15 (the year 1950); therefore John's age will lie between 15 and 25. His birthdate must have been in the fifties, say $1950 + x$ where x is a digit. His age will reach $25 - x$ in 1975. Hence

$$25 - x = 15 + x, \quad \text{giving } x = 5.$$

John was born in 1955.